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Canonical Form Algorithm and Separating Systems for Two-Dimensional Algebras

Ahmed, H.[1], Bekbaev, U.[2], and Rakhimov, I. *[3,4]

[1]*Department of Mathematics, Faculty of Science, Taiz University, Taiz, Yemen*
[2]*Department of Mathematical and Natural Sciences, Turin Polytechnic University in Tashkent, Uzbekistan*
[3]*Faculty of Computer and Mathematical Sciences, UiTM, Shah Alam, Malaysia*
[4]*V.I. Romanovski Institute of Mathematics, Uzbekistan Academy of Sciences, Uzbekistan*

E-mail: isamiddin@uitm.edu.my
**Corresponding author*

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Abstract

For a field $\mathbb{F}$, where any polynomial of degree two and three possess a root, we give an algorithm transforming any two-dimensional algebra over $\mathbb{F}$ to a "canonical form" in terms of structure constants of the algebra. A system of invariant conditions in form of equalities and non-equalities (inequalities) separating non-isomorphic two-dimensional algebras is given.

Keywords: algebra; invariant; matrix of structure constants; canonical form.

1 Introduction

Classification of mathematical structures, for example algebras, up to isomorphism is one of the important problems in Mathematics. The solution to the problem, in general, is very hard and in algebras case there are particular solutions as the classifications of various subclasses of algebras with polynomial identities, such as simple associative, simple(semi-simple) Lie, simple Jordan, simple Malcev etc. In a fixed n-dimensional case the existing classifications capture a small part of all n-dimensional algebras due to polynomial identity form conditions.

Apart of its importance in algebra the problem of classification of G-orbits has applications in the theory of continuous and discrete dynamical systems (see [7, 8, 11] (These papers deal with $n = 2$ case under the restriction $Char\mathbb{F} \neq 3$ supplementary to $Char\mathbb{F} \neq 2$).

The authors in [1] gave a complete classification, up to isomorphism, of all nontrivial two dimensional algebras in terms of their matrices of structure constants (MSC) over any field $\mathbb{F}$ where every second and third order polynomial possesses a root. However, in [1] no algorithm to transform any two-dimensional algebra to the canonical form has been given. The goal of the present paper is two-fold. The first one is to give such an algorithm. The second provides a system of invariant conditions consisting of equalities and non-equalities (inequalities) of rational functions separating non-isomorphic two-dimensional algebras.

Thereby we complete the solution of the classification problem for all two-dimensional algebras over the field mentioned above. We reconfirm the results of [1] and present a system of invariant conditions, consisting of equalities and non-equalities(inequalities) of invariant rational functions in terms of MSC. The main result is formulated in an algorithmic form following that one can find the canonical form of any two-dimensional algebra given by its MSC.

For the studies on the classification and other problems related to two-dimensional algebras we refer to [2, 3, 4, 8, 9, 10].

2 Preliminaries

Let for a while $\mathbb{F}$ be an algebraically closed field and G be an algebraic group acting on an irreducible algebraic variety X :

$$G \times X \longrightarrow X \quad (g, x) \longmapsto g \cdot x.$$

This action generates an action of G on the field of rational function $\mathbb{F}(X)$ of X :

$$(g \cdot f)(x) = f(g^{-1} \cdot x), \text{ for all } f \in \mathbb{F}(X).$$

The set of all invariant rational functions $\mathbb{F}(X)^G$ forms a subfield of $\mathbb{F}(X)$. Being a subfield of a finitely generated field $\mathbb{F}(X)$ the field $\mathbb{F}(X)^G$ has a finitely many generators.

It is said that an invariant function f separates orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ if it is defined on the both orbits and takes on them different values. And it is said that a set of invariant functions S separates orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ if S contains an element separating these two orbits. We say that a set of invariantes S separates generic orbits if there exists a non-empty open subset X_0 of X such that the set S separates orbits in X_0. The existence such a set of invariants S for the case where an algebraic group G acts on an irreducible algebraic variety X has been given by M. Rosenlicht (see [12] and [6] for a modern version). However, the description and the number of its elements depends on the representation considered. The following result holds true ([13]).

Lemma 2.1. *If a finite set* $S \subset \mathbb{F}(X)^G$ *separates orbits of points in general position then* S *generates* $\mathbb{F}(X)^G$.

M. Rosenlicht also has shown that orbits in general position in X can be separated by rational invariants.

We say that $f \in \mathbb{F}(X)$ has invariant property over an orbit $\mathcal{O}$ if it is defined at any element of $\mathcal{O}$, $f|_{\mathcal{O}}$ is constant or it does not vanish at any element of $\mathcal{O}$ that is $f|_{\mathcal{O}}$ does not vanish at all. Let us denote by $Inv(\mathcal{O})$ the set of all such functions.

It is said that $f \in \mathbb{F}(X)$ separates orbits $\mathcal{O}_1$, $\mathcal{O}_2$ if $f \in Inv(\mathcal{O}_1) \cap Inv(\mathcal{O}_2)$ and $f|_{\mathcal{O}_1}$, $f|_{\mathcal{O}_2}$ are constant functions with different values or one of them is identically zero and the second one does not vanish at all.

It is said say that a finite set S of elements of $\mathbb{F}(X)$ separates a given set of orbits if for any two orbits there exists an element of S which separates these two orbits. If one looks for a finite set of functions separating all, not only generic, orbits the one need to look for a finite set of above mentioned functions with invariant properties. In this sense one should expect that in most, if not in all, cases there exists a finite set of functions separating all orbits.

Further we need the following result obtained by U. Bekbaev in [5]. We give here the result in a slightly different form. Let n be a natural number, G be a group, $\mathbb{F}$ be any field, $\tau : G \to GL(n, \mathbb{F})$ be a homomorphism of groups and $M \subset \mathbb{F}^n$ be a τ-invariant subset, that is $\tau(g)u \in M$ whenever $u \in M$ and $g \in G$. We write $u \simeq^\tau v$ if $\tau(g)u = v$ for some $g \in G$.

Lemma 2.2. *If there exists* $m \in \mathbb{N}$*, a map* $P : M \to GL(m, \mathbb{F})$ *and a homomorphism* $\tau' : G' \to GL(n, \mathbb{F})$ *such that*

$$\tau'(P(\tau(g)u)) = \tau'(P(u))\tau(g^{-1}) \text{ whenever } u \in M,\ g \in G$$

and $v, w \in M$ *are any two elements then* $v \simeq^G w$ *if and only if*

$$\tau'(P(v))v = \tau'(P(w))w \text{ and } \tau'(P(w)^{-1}P(v)) \in \tau(G),$$

where G' *stands for the subgroup of* $GL(m, \mathbb{F})$ *generated by* $\{P(u) : u \in M\}$.

We consider all the above said in the following situation. Let V be an n-dimensional vector space over $\mathbb{F}$. The set of n-dimensional algebras on V we denote by $Alg_n(\mathbb{F})$. If we fix a basis $\mathbf{e} = \{e_1, e_2, ..., e_n\}$ then $Alg_n(\mathbb{F})$ can be identified with $Mat(n \times n^2, \mathbb{F})$. Putting in correspondence to an element $\mathbb{A} \in Alg_n(\mathbb{F})$ defined by a bilinear binary operation $V \times V \longrightarrow V$ we obtain a well-defined bijection between $Alg_n(\mathbb{F})$ and the set of all bilinear maps on V. The action of $GL_n(\mathbb{F})$ on $Mat(n \times n^2, \mathbb{F})$ we define by the formula

$$(g \cdot A) = gA(g^{-1})^{\otimes 2} \text{ for all } g \in GL_n(\mathbb{F}) \text{ and } A \in Mat(n \times n^2, \mathbb{F}).$$

3 Main Result

Now and onward, the field $\mathbb{F}$ is fixed such that in $\mathbb{F}$ any polynomial of degree two and three over $\mathbb{F}$ possess a root. If $\mathbb{A}$ is a two-dimensional algebra over $\mathbb{F}$ and $\mathbf{e} = (e_1, e_2)$ is a fixed basis we denote by

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \in Mat(2 \times 4, \mathbb{F});$$

its matrix of structure constants (MSC) with respect to this basis, i.e.,

$$e_1e_1 = \alpha_1 e_1 + \beta_1 e_2,\ e_1e_2 = \alpha_2 e_1 + \beta_2 e_2,\ e_2e_1 = \alpha_3 e_1 + \beta_3 e_2,\ e_2e_2 = \alpha_4 e_1 + \beta_4 e_2.$$

Further it is assumed that a basis $\mathbf{e}$ is fixed, we don't make difference between an algebra $\mathbb{A}$ and its MSC A. Denote

$$Tr_1(A) = (\alpha_1 + \beta_3, \alpha_2 + \beta_4),\ Tr_2(A) = (\alpha_1 + \beta_2, \alpha_3 + \beta_4),\ \Delta(A) = \det \begin{pmatrix} \alpha_1 + \beta_3 & \alpha_2 + \beta_4 \\ \alpha_1 + \beta_2 & \alpha_3 + \beta_4 \end{pmatrix}.$$

If $\Delta(A) = 0$ and $Tr_1(A) \neq 0$, that is $\lambda Tr_1(A) = Tr_2(A)$, then $\lambda = \lambda(A) = \frac{\alpha_1+\beta_2}{\alpha_1+\beta_3}$ in $\alpha_1 + \beta_3 \neq 0$ case and $\lambda = \lambda(A) = \frac{\alpha_3+\beta_4}{\alpha_2+\beta_4}$ if $\alpha_2 + \beta_4 \neq 0$ case. Note that if $\alpha_1 + \beta_3 \neq 0$ and $\alpha_2 + \beta_4 \neq 0$ then $\frac{\alpha_1+\beta_2}{\alpha_1+\beta_3} = \frac{\alpha_3+\beta_4}{\alpha_2+\beta_4}$.

If $\mathbf{e}' = (e'_1, e'_2)$ is another basis of $\mathbb{A}$, $\mathbf{e}'g = \mathbf{e}$ with $g \in G = GL(2; \mathbb{F})$, A' is MSC of $\mathbb{A}$ with respect to $\mathbf{e}'$ then

$$A' = gA(g^{-1})^{\otimes 2}$$

is valid and $Tr_i(A') = Tr_i(A)g^{-1}$, $i = 1, 2$, where $\otimes$ stands for corresponding tensor(Kronecker) product of matrices (see [1]). Thus one comes to the following definition of isomorphism of algebras in terms their MSCs.

Definition 3.1. *Two 2-dimensional algebras $\mathbb{A}$, $\mathbb{B}$ over $\mathbb{F}$, given by their matrices of structure constants A, B, are said to be isomorphic, notation $\mathbb{A} \simeq \mathbb{B}$ or $A \simeq B$, if $B = gA(g^{-1})^{\otimes 2}$ holds true for some $g \in GL(2; \mathbb{F})$.*

Let $\mathbb{A}$ be any two-dimensional algebra over $\mathbb{F}$ and A be its MSC with respect to $\mathbf{e} = (e_1, e_2)$. As mentioned above one of the aim of the paper is to transfer MSC of any two-dimensional algebra over $\mathbb{F}$ to a so-called "canonical" form. Further for the entries of the canonical MSC we keep use the same letters α, β, with subindices "i, c", and provide expressions for them in terms of the original entries of A. The canonical matrices are equated to A_i, where $i = 1, 2, ..., 12$, to indicate the correspondence with the results of [1]. We use the same notation A_i (some of them with parameters), where $i = 1, 2, ..., 12$, in [1] for the canonical MSC.

In $Char(\mathbb{F}) \neq 2, 3$ the theorem below gives an algorithm for finding the canonical form of any two-dimensional algebra given by its MCS A. So, in general, the system of non-constant entries of the canonical matrix is a system of invariant rational functions separating any two non-isomorphic 2-dimensional algebras from the corresponding class of algebras (see Theorem 3.1).

Theorem 3.1. *Let $Char(\mathbb{F}) \neq 2, 3$.*

A. If $\Delta(A) \neq 0$ then

$$A \simeq \begin{pmatrix} \alpha_{1,c} & \alpha_{2,c} & \alpha_{2,c}+1 & \alpha_{4,c} \\ \beta_{1,c} & -\alpha_{1,c} & 1-\alpha_{1,c} & -\alpha_{2,c} \end{pmatrix} = A_1,$$

where

$$\begin{array}{rcl}
\alpha_{1,c} & = & \frac{1}{\Delta(A)^2}(-\alpha_1^2\alpha_2\alpha_3+\alpha_1^3\alpha_4+\alpha_2\alpha_3^2\beta_1-2\alpha_1\alpha_2\alpha_3\beta_2-\alpha_1\alpha_3^2\beta_2+2\alpha_1^2\alpha_4\beta_2-\alpha_2\alpha_3\beta_2^2\\
& & +\alpha_1\alpha_4\beta_2^2-2\alpha_1\alpha_2\alpha_3\beta_3+\alpha_1^2\alpha_4\beta_3-2\alpha_2\alpha_3\beta_2\beta_3-\alpha_3^2\beta_2\beta_3+2\alpha_1\alpha_4\beta_2\beta_3+\alpha_4\beta_2^2\beta_3\\
& & +\alpha_1^2\alpha_3\beta_4+2\alpha_2\alpha_3\beta_1\beta_4+\alpha_3^2\beta_1\beta_4-2\alpha_1\alpha_3\beta_2\beta_4-\alpha_3\beta_2^2\beta_4-2\alpha_1\alpha_2\beta_3\beta_4\\
& & -2(\alpha_2+\alpha_3)\beta_2\beta_3\beta_4+2\alpha_1^2\beta_4^2+\alpha_2\beta_1\beta_4^2+2\alpha_3\beta_1\beta_4^2+\alpha_1\beta_2\beta_4^2-\beta_2\beta_3\beta_4^2+\beta_1\beta_4^3),\\
\alpha_{2,c} & = & \frac{1}{\Delta(A)^2}(\alpha_1^2\alpha_2\alpha_3-\alpha_1^3\alpha_4-\alpha_2^2\alpha_3\beta_1+2\alpha_1\alpha_2\alpha_3\beta_2-\alpha_1^2\alpha_4\beta_2+\alpha_1\alpha_2^2\beta_3+2\alpha_1\alpha_2\alpha_3\beta_3\\
& & -2\alpha_1^2\alpha_4\beta_3+\alpha_2^2\beta_2\beta_3+2\alpha_2\alpha_3\beta_2\beta_3-2\alpha_1\alpha_4\beta_2\beta_3+\alpha_2\alpha_3\beta_3^2-\alpha_1\alpha_4\beta_3^2-\alpha_4\beta_2\beta_3^2\\
& & -\alpha_1^2\alpha_2\beta_4-\alpha_2^2\beta_1\beta_4-2\alpha_2\alpha_3\beta_1\beta_4+2\alpha_1\alpha_3\beta_2\beta_4+2\alpha_1\alpha_2\beta_3\beta_4+2\alpha_2\beta_2\beta_3\beta_4\\
& & +2\alpha_3\beta_2\beta_3\beta_4+\alpha_2\beta_3^2\beta_4-2\alpha_1^2\beta_4^2-2\alpha_2\beta_1\beta_4^2-\alpha_3\beta_1\beta_4^2-\alpha_1\beta_3\beta_4^2+\beta_2\beta_3\beta_4^2-\beta_1\beta_4^3),\\
\alpha_{4,c} & = & \frac{1}{\Delta(A)^2}(-\alpha_1^2\alpha_2\alpha_3+\alpha_1^3\alpha_4+\alpha_2^3\beta_1-\alpha_1\alpha_2^2\beta_2-2\alpha_1\alpha_2^2\beta_3-2\alpha_1\alpha_2\alpha_3\beta_3+3\alpha_1^2\alpha_4\beta_3\\
& & -\alpha_2^2\beta_2\beta_3-2\alpha_2^2\beta_3^2-\alpha_2\alpha_3\beta_3^2+3\alpha_1\alpha_4\beta_3^2+\alpha_4\beta_3^3+2\alpha_1^2\alpha_2\beta_4-\alpha_1^2\alpha_3\beta_4+3\alpha_2^2\beta_1\beta_4\\
& & -2\alpha_1\alpha_2\beta_2\beta_4-2\alpha_1\alpha_3\beta_3\beta_4-2\alpha_2\beta_2\beta_3\beta_4-2\alpha_2\beta_3^2\beta_4-\alpha_3\beta_3^2\beta_4+2\alpha_1^2\beta_4^2+3\alpha_2\beta_1\beta_4^2\\
& & -\alpha_1\beta_2\beta_4^2+2\alpha_1\beta_3\beta_4^2-\beta_2\beta_3\beta_4^2+\beta_1\beta_4^3),\\
\beta_{1,c} & = & \frac{1}{\Delta(A)^2}(-\alpha_1^2\alpha_2\alpha_3+\alpha_1^3\alpha_4+\alpha_3^3\beta_1-2\alpha_1\alpha_2\alpha_3\beta_2-2\alpha_1\alpha_3^2\beta_2+3\alpha_1^2\alpha_4\beta_2-\alpha_2\alpha_3\beta_2^2\\
& & -2\alpha_3^2\beta_2^2+3\alpha_1\alpha_4\beta_2^2+\alpha_4\beta_2^3-\alpha_1\alpha_3^2\beta_3-\alpha_3^2\beta_2\beta_3-\alpha_1^2\alpha_2\beta_4+2\alpha_1^2\alpha_3\beta_4+3\alpha_3^2\beta_1\beta_4\\
& & -2\alpha_1\alpha_2\beta_2\beta_4-\alpha_2\beta_2^2\beta_4-2\alpha_3\beta_2^2\beta_4-2\alpha_1\alpha_3\beta_3\beta_4-2\alpha_3\beta_2\beta_3\beta_4+2\alpha_1^2\beta_4^2+3\alpha_3\beta_1\beta_4^2\\
& & +2\alpha_1\beta_2\beta_4^2-\alpha_1\beta_3\beta_4^2-\beta_2\beta_3\beta_4^2+\beta_1\beta_4^3)
\end{array}$$

and for such algebras the system of functions $\alpha_{1,c},\ \alpha_{2,c}, \alpha_{4,c},\ \beta_{1,c}$ *is a separating system of invariants.*

B. *If* $\Delta(A)=0$*,*

- $\alpha_1+\beta_3\neq 0$ ($\alpha_2+\beta_4\neq 0$)
  - $\alpha_{4,2}(A)=-2\alpha_1^2\beta_4^2\lambda-4\alpha_1^2\alpha_2\beta_4\lambda-4\alpha_1\beta_3\beta_4^2\lambda-4\alpha_1\alpha_2^2\beta_3\lambda-8\alpha_1\alpha_2\beta_3\beta_4\lambda$ $-2\alpha_2^2\beta_3^2\lambda-4\alpha_2\beta_3^2\beta_4\lambda+4\alpha_1^2\beta_4^2+3\alpha_4\alpha_1^2\beta_3+5\alpha_1^2\alpha_2\beta_4+3\alpha_1\alpha_4\beta_3^2$ $+5\alpha_1\beta_3\beta_4^2-\alpha_1\alpha_2^2\beta_3+4\alpha_1\alpha_2\beta_3\beta_4+\alpha_4\beta_3^3-2\alpha_2^2\beta_3^2+3\alpha_2\beta_1\beta_4^2+\alpha_2^3\beta_1$ $-\alpha_2\beta_3^2\beta_4+3\alpha_2^2\beta_1\beta_4-2\alpha_1^2\alpha_2^2\lambda+\alpha_1^3\alpha_4+\alpha_1^2\alpha_2^2-2\beta_3^2\beta_4^2\lambda+\beta_1\beta_4^3$ $+\beta_3^2\beta_4^2\neq 0$
    then

$$A\simeq\begin{pmatrix}\alpha_{1,c} & 0 & 0 & 1\\ \beta_{1,c} & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 0\end{pmatrix}\simeq\begin{pmatrix}\alpha_{1,c} & 0 & 0 & 1\\ -\beta_{1,c} & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 0\end{pmatrix}=A_2,$$

where

$$\begin{array}{rcl}
\alpha_{1,c} & = & \frac{1}{(\alpha_1+\beta_3)^2}\Big(\alpha_1\beta_3+\alpha_2\beta_1+\alpha_1^2+\beta_1\beta_4-(\alpha_1^2\beta_4-\alpha_1\alpha_2(\beta_2+\beta_3)-\alpha_1\beta_2\beta_4\\
& & +\alpha_1\beta_3\beta_4-\alpha_2\beta_3(\beta_2+\beta_3)+\alpha_2^2\beta_1+2\alpha_2\beta_1\beta_4+\beta_1\beta_4^2-\beta_2\beta_3\beta_4)^2/\big(2\alpha_1^2\beta_4^2\\
& & +3\alpha_1^2\alpha_4\beta_3+\alpha_1^2\alpha_2\beta_4+3\alpha_1\alpha_4\beta_3^2-2\alpha_1\beta_2\beta_4^2+3\alpha_1\beta_3\beta_4^2-2\alpha_1\alpha_2^2\beta_2\\
& & -3\alpha_1\alpha_2^2\beta_3-4\alpha_1\alpha_2\beta_2\beta_4+\alpha_4\beta_3^3-2\alpha_2^2\beta_3^2+3\alpha_2\beta_1\beta_4^2+\alpha_2^3\beta_1-2\alpha_2^2\beta_2\beta_3\\
& & -\alpha_2\beta_3^2\beta_4+3\alpha_2^2\beta_1\beta_4-4\alpha_2\beta_2\beta_3\beta_4+\alpha_1^3\alpha_4-\alpha_1^2\alpha_2^2+\beta_1\beta_4^3\\
& & +\beta_3^2\beta_4^2-2\beta_2\beta_3\beta_4^2)\Big),\\
\beta_{1,c} & = & \Big(\alpha_4\beta_4\alpha_1^5+\alpha_2\beta_4^2\alpha_1^4-\alpha_4^2\beta_1\alpha_1^4-\alpha_2\alpha_4\beta_2\alpha_1^4-\alpha_2\alpha_4\beta_3\alpha_1^4-\alpha_2^2\beta_4\alpha_1^4\\
& & -2\alpha_4\beta_2\beta_4\alpha_1^4+2\alpha_4\beta_3\beta_4\alpha_1^4-\beta_3\beta_4^3\alpha_1^3+\alpha_2\alpha_4\beta_2^2\alpha_1^3-2\alpha_2\alpha_4\beta_3^2\alpha_1^3\\
& & -2\alpha_2\beta_2\beta_4^2\alpha_1^3+3\alpha_2\beta_3\beta_4^2\alpha_1^3+3\alpha_2^2\alpha_4\beta_1\alpha_1^3+\alpha_2^3\beta_2\alpha_1^3+\alpha_2^3\beta_3\alpha_1^3\\
& & -4\alpha_4^2\beta_1\beta_3\alpha_1^3-\alpha_2\alpha_4\beta_2\beta_3\alpha_1^3+\alpha_4\beta_2^2\beta_4\alpha_1^3+3\alpha_2\alpha_4\beta_1\beta_4\alpha_1^3-\alpha_2^2\beta_2\beta_4\alpha_1^3\\
& & -3\alpha_2^2\beta_3\beta_4\alpha_1^3-5\alpha_4\beta_2\beta_3\beta_4\alpha_1^3-\beta_1\beta_4^4\alpha_1^2-2\beta_3^2\beta_4^3\alpha_1^2+\alpha_2\beta_1\beta_4^3\alpha_1^2\\
& & +2\beta_2\beta_3\beta_4^3\alpha_1^2+\alpha_2^3\beta_2^2\alpha_1^2+2\alpha_2^3\beta_3^2\alpha_1^2-6\alpha_4^2\beta_1\beta_3^2\alpha_1^2+3\alpha_2\alpha_4\beta_2\beta_3^2\alpha_1^2\\
& & +\alpha_2\beta_2^2\beta_4^2\alpha_1^2+4\alpha_2\beta_3^2\beta_4^2\alpha_1^2+3\alpha_2^2\beta_1\beta_4^2\alpha_1^2-3\alpha_4\beta_1\beta_3\beta_4^2\alpha_1^2-2\alpha_2\beta_2\beta_3\beta_4^2\alpha_1^2\\
& & -2\alpha_2^4\beta_1\alpha_1^2+3\alpha_2\alpha_4\beta_2^2\beta_3\alpha_1^2+6\alpha_2^2\alpha_4\beta_1\beta_3\alpha_1^2+3\alpha_2^3\beta_2\beta_3\alpha_1^2-2\alpha_4\beta_3^3\beta_4\alpha_1^2
\end{array}$$

$$
\begin{aligned}
&+2\alpha_2^2\beta_2^2\beta_4\alpha_1^2-4\alpha_2^2\beta_3^2\beta_4\alpha_1^2-3\alpha_4\beta_2\beta_3^2\beta_4\alpha_1^2-\alpha_2^3\beta_1\beta_4\alpha_1^2+3\alpha_4\beta_2^2\beta_3\beta_4\alpha_1^2\\
&+\alpha_1^2\alpha_2\beta_3\beta_4(3\alpha_4\beta_1-\alpha_2\beta_2)+2\alpha_2\alpha_4\beta_3^4\alpha_1+2\beta_1\beta_2\beta_4^4\alpha_1-3\beta_1\beta_3\beta_4^4\alpha_1\\
&+\alpha_1\beta_3^3(\alpha_2^3-4\alpha_4^2\beta_1)+5\alpha_2\alpha_4\beta_2\beta_3^3\alpha_1-\alpha_1\beta_4^3(\beta_3^3-\alpha_4\beta_1^2-3\beta_2\beta_3^2)\\
&+\alpha_1\beta_2\beta_4^3(3\alpha_2\beta_1-\beta_2\beta_3)+\alpha_2^3\alpha_4\beta_1^2\alpha_1+3\alpha_2\alpha_4\beta_2^2\beta_3^2\alpha_1+3\alpha_2^2\alpha_4\beta_1\beta_3^2\alpha_1\\
&+2\alpha_2^3\beta_2\beta_3^2\alpha_1+3\alpha_1\alpha_2\beta_4^2(\beta_3^3+\alpha_4\beta_1^2)-6\alpha_4\beta_1\beta_3^2\beta_4^2\alpha_1-\alpha_2\beta_2\beta_3^2\beta_4^2\alpha_1\\
&-3\alpha_2^2\beta_1\beta_2\beta_4^2\alpha_1-\alpha_1\alpha_2\beta_2^2\beta_3(\beta_4^2-\alpha_2^2)+6\alpha_2^2\beta_1\beta_3\beta_4^2\alpha_1-3\alpha_2^4\beta_1\beta_2\alpha_1\\
&-3\alpha_2^4\beta_1\beta_3\alpha_1-\alpha_4\beta_3^4\beta_4\alpha_1-3\alpha_2^2\beta_3^3\beta_4\alpha_1+\alpha_4\beta_2\beta_3^3\beta_4\alpha_1+3\alpha_2^2\alpha_4\beta_1^2\beta_4\alpha_1\\
&+3\alpha_4\beta_2^2\beta_3^2\beta_4\alpha_1-3\alpha_2\alpha_4\beta_1\beta_3^2\beta_4\alpha_1-2\alpha_2^2\beta_2\beta_3^2\beta_4\alpha_1-7\alpha_2^3\beta_1\beta_2\beta_4\alpha_1\\
&+\alpha_2^2\beta_2^2\beta_3\beta_4\alpha_1+\alpha_2\alpha_4\beta_3^5-\alpha_4^2\beta_1\beta_3^4+2\alpha_2\alpha_4\beta_2\beta_3^4-\beta_1\beta_2^2\beta_4^4-2\beta_1\beta_3^2\beta_4^4\\
&+3\beta_1\beta_2\beta_3\beta_4^4+\alpha_2\alpha_4\beta_2^2\beta_3^3+\beta_2\beta_3^3\beta_4^3-4\alpha_2\beta_1\beta_2^2\beta_4^3-\beta_2^2\beta_3^2\beta_4^3-\alpha_2\beta_1\beta_3^2\beta_4^3\\
&+\alpha_4\beta_1^2\beta_3\beta_4^3+7\alpha_2\beta_1\beta_2\beta_3\beta_4^3-\alpha_2^4\beta_1\beta_2^2-\alpha_2^4\beta_1\beta_3^2+\alpha_2\beta_3^4\beta_4^2-3\alpha_4\beta_1\beta_3^3\beta_4^2\\
&-\alpha_2\beta_2\beta_3^3\beta_4^2-6\alpha_2^2\beta_1\beta_2^2\beta_4^2-2\alpha_2\beta_2^2\beta_3^2\beta_4^2+3\alpha_2^2\beta_1\beta_3^2\beta_4^2+3\alpha_2\alpha_4\beta_1^2\beta_3\beta_4^2\\
&+\alpha_2^2\beta_1\beta_2\beta_3(3\beta_4^2-2\alpha_2^2)+\alpha_2^3\alpha_4\beta_1^2\beta_3-\alpha_2^2\beta_3^4\beta_4+\alpha_4\beta_2\beta_3^4\beta_4+\alpha_4\beta_2^2\beta_3^3\beta_4\\
&-3\alpha_2\alpha_4\beta_1\beta_3^3\beta_4-2\alpha_2^2\beta_2\beta_3^3\beta_4-4\alpha_2^3\beta_1\beta_2^2\beta_4-\alpha_2^2\beta_2^2\beta_3^2\beta_4+\alpha_2^3\beta_1\beta_3^2\beta_4\\
&+3\alpha_2^2\beta_1\beta_3\beta_4(\alpha_4\beta_1-\alpha_2\beta_2)\Bigg)\Bigg/\Bigg((\alpha_1+\beta_3)(2\alpha_1^2\beta_4^2+3\alpha_4\alpha_1^2\beta_3+\alpha_2\alpha_1^2\beta_4\\
&+\alpha_4\beta_3^2(3\alpha_1+\beta_3)-\alpha_1\beta_4^2(2\beta_2-3\beta_3)-\alpha_1\alpha_2^2(2\beta_2+3\beta_3)-4\alpha_1\alpha_2\beta_2\beta_4\\
&-2\alpha_2^2\beta_3(\beta_3+\beta_2)+3\alpha_2\beta_1\beta_4^2+\alpha_2^3\beta_1-\alpha_2\beta_3^2\beta_4+3\alpha_2^2\beta_1\beta_4-4\alpha_2\beta_2\beta_3\beta_4\\
&+\alpha_4\alpha_1^3-\alpha_1^2\alpha_2^2+\beta_1\beta_4^3+\beta_3^2\beta_4^2-2\beta_2\beta_3\beta_4^2)^{3/2}\Bigg).
\end{aligned}
$$

Respectively,

$$
\begin{aligned}
\alpha_{1,c} \;=\;& \tfrac{1}{(\alpha_2+\beta_4)^2}\Bigg(\alpha_2\beta_4+\alpha_4\beta_3+\alpha_1\alpha_4+\beta_4^2-\big(\alpha_1\beta_4^2+2\alpha_4\alpha_1\beta_3+\alpha_2\alpha_1\beta_4\\
&-\alpha_3\alpha_1\beta_4+\alpha_4\beta_3^2-\alpha_2^2\beta_3-\alpha_2\alpha_3\beta_3-\alpha_2\beta_3\beta_4-\alpha_3\beta_3\beta_4+\alpha_4\alpha_1^2\\
&-\alpha_1\alpha_2\alpha_3\big)^2\big/\big(2\alpha_1^2\beta_4^2+3\alpha_4\alpha_1^2\beta_3+3\alpha_2\alpha_1^2\beta_4-2\alpha_3\alpha_1^2\beta_4+3\alpha_4\alpha_1\beta_3^2\\
&+\alpha_1\beta_3\beta_4^2-\alpha_2^2\alpha_1\beta_3-4\alpha_2\alpha_3\alpha_1\beta_3-4\alpha_3\alpha_1\beta_3\beta_4+\alpha_4\beta_3^3-2\alpha_2^2\beta_3^2\\
&-2\alpha_2\alpha_3\beta_3^2+3\alpha_2\beta_1\beta_4^2+\alpha_2^3\beta_1-3\alpha_2\beta_3^2\beta_4-2\alpha_3\beta_3^2\beta_4+3\alpha_2^2\beta_1\beta_4\\
&+\alpha_4\alpha_1^3+\alpha_2^2\alpha_1^2-2\alpha_2\alpha_3\alpha_1^2+\beta_1\beta_4^3-\beta_3^2\beta_4^2)\Bigg),\\
\beta_{1,c} \;=\;& \Bigg(\beta_1\beta_3\alpha_2^5-\alpha_4\beta_1^2\alpha_2^4-\alpha_1\beta_3^2\alpha_2^4+\alpha_1\alpha_3\beta_1\alpha_2^4+\alpha_1^2\beta_3\alpha_2^4+2\alpha_3\beta_1\beta_3\alpha_2^4\\
&-\alpha_1\beta_1\beta_4\alpha_2^4+2\beta_1\beta_3\beta_4\alpha_2^4-2\alpha_1\alpha_3\beta_3^2\alpha_2^3-2\alpha_1\beta_1\beta_4^2\alpha_2^3+\alpha_1^3\alpha_3\alpha_2^3\\
&+\alpha_1\alpha_3^2\beta_1\alpha_2^3-3\alpha_1^2\alpha_4\beta_1\alpha_2^3-\alpha_1^2\alpha_3\beta_3\alpha_2^3+\alpha_3^2\beta_1\beta_3\alpha_2^3-3\alpha_1\alpha_4\beta_1\beta_3\alpha_2^3\\
&-\alpha_1^3\beta_4\alpha_2^3+\beta_3^3\beta_4\alpha_2^3-4\alpha_4\beta_1^2\beta_4\alpha_2^3-3\alpha_1\beta_3^2\beta_4\alpha_2^3+\alpha_1\alpha_3\beta_1\beta_4\alpha_2^3\\
&+3\alpha_1^2\beta_3\beta_4\alpha_2^3+5\alpha_3\beta_1\beta_3\beta_4\alpha_2^3-\alpha_4\beta_3^4\alpha_2^2+\alpha_1\alpha_4\beta_3^3\alpha_2^2-2\beta_1\beta_3\beta_4^3\alpha_2^2\\
&-\alpha_1^3\alpha_3^2\alpha_2^2-\alpha_1\alpha_3^2\beta_3^2\alpha_2^2+3\alpha_1^2\alpha_4\beta_3^2\alpha_2^2-2\alpha_1^3\beta_4^2\alpha_2^2+2\beta_3^3\beta_4^2\alpha_2^2-6\alpha_4\beta_1^2\beta_4^2\alpha_2^2\\
&-4\alpha_1\beta_3^2\beta_4^2\alpha_2^2-3\alpha_1\alpha_3\beta_1\beta_4^2\alpha_2^2+4\alpha_1^2\beta_3\beta_4^2\alpha_2^2+3\alpha_3\beta_1\beta_3\beta_4^2\alpha_2^2-2\alpha_1^4\alpha_4\alpha_2^2\\
&-2\alpha_1^2\alpha_3^2\beta_3\alpha_2^2-\alpha_1^3\alpha_4\beta_3\alpha_2^2+2\alpha_3\beta_3^3\beta_4\alpha_2^2-2\alpha_1\alpha_3\beta_3^2\beta_4\alpha_2^2+3\alpha_4\beta_1\beta_3^2\beta_4\alpha_2^2\\
&+3\alpha_1^3\alpha_3\beta_4\alpha_2^2+3\alpha_1\alpha_3^2\beta_1\beta_4\alpha_2^2-6\alpha_1^2\alpha_4\beta_1\beta_4\alpha_2^2-\alpha_1^2\alpha_3\beta_3\beta_4\alpha_2^2\\
&+3\alpha_2^2\beta_1\beta_3\beta_4(\alpha_3^2-\alpha_1\alpha_4)-2\alpha_3\alpha_4\beta_3^4\alpha_2+2\alpha_1\beta_1\beta_4^4\alpha_2-\beta_1\beta_3\beta_4^4\alpha_2\\
&-3\alpha_1\alpha_3\alpha_4\beta_3^3\alpha_2+\alpha_4^2\beta_1\beta_3^3\alpha_2-\alpha_1^3\beta_4^3\alpha_2+\beta_3^3\beta_4^3\alpha_2-4\alpha_4\beta_1^2\beta_4^3\alpha_2\\
&-\alpha_1\alpha_2\beta_4^3(3\beta_3^2+5\alpha_3\beta_1)+3\alpha_1^2\beta_3\beta_4^3\alpha_2-\alpha_3\beta_1\beta_3\beta_4^3\alpha_2+3\alpha_1^2\alpha_3\alpha_4\beta_3^2\alpha_2\\
&+3\alpha_2\beta_3^2(\alpha_1\alpha_4^2\beta_1+\alpha_3\beta_3\beta_4^2)-\alpha_1\alpha_3\beta_3^2\beta_4^2\alpha_2+6\alpha_4\beta_1\beta_3^2\beta_4^2\alpha_2+2\alpha_1^3\alpha_3\beta_4^2\alpha_2\\
&+3\alpha_1\alpha_3^2\beta_1\beta_4^2\alpha_2-3\alpha_1^2\alpha_4\beta_1\beta_4^2\alpha_2-2\alpha_1^2\alpha_3\beta_3\beta_4^2\alpha_2+3\alpha_3^2\beta_1\beta_3\beta_4^2\alpha_2\\
&+3\alpha_1\alpha_4\beta_1\beta_3\beta_4^2\alpha_2+3\alpha_1^4\alpha_3\alpha_4\alpha_2+\alpha_1^3\alpha_4^2\beta_1\alpha_2+7\alpha_1^3\alpha_3\alpha_4\beta_3\alpha_2\\
&+3\alpha_1^2\alpha_4^2\beta_1\beta_3\alpha_2-3\alpha_4\beta_3^4\beta_4\alpha_2+\alpha_3^2\beta_3^3\beta_4\alpha_2-\alpha_1^3\alpha_3^2\beta_4\alpha_2+\alpha_1\alpha_3^2\beta_3^2\beta_4\alpha_2\\
&+3\alpha_1^2\alpha_2\alpha_4\beta_4(2\beta_3^2-\alpha_1^2)-\alpha_1^2\alpha_3^2\beta_3\beta_4\alpha_2+\alpha_1\beta_1\beta_4^5-\alpha_3^2\alpha_4\beta_3^4-\alpha_4\beta_1^2\beta_4^4
\end{aligned}
$$

$$-\alpha_1\beta_3^2\beta_4^4 - 2\alpha_1\alpha_3\beta_1\beta_4^4 + \alpha_1^2\beta_3\beta_4^4 - \alpha_3\beta_1\beta_3\beta_4^4 - 4\alpha_1\alpha_3^2\alpha_4\beta_3^3 + \alpha_3\beta_3^3\beta_4^3$$
$$-\alpha_1\alpha_3\beta_3^2\beta_4^3 + 3\alpha_4\beta_1\beta_3^2\beta_4^3 + \alpha_1\alpha_3^2\beta_1\beta_4^3 - 2\alpha_1^2\alpha_3\beta_3\beta_4^3 + \alpha_3^2\beta_1\beta_3\beta_4^3$$
$$+3\alpha_1\alpha_4\beta_3(\beta_1\beta_4^3 - 2\alpha_1\alpha_3^2\beta_3) - 2\beta_3^2\beta_4^2(\alpha_4\beta_3^2 - \alpha_1\alpha_3^2) + \beta_3^3\beta_4^2(\alpha_3^3 - \alpha_1\alpha_4)$$
$$+3\alpha_1^2\alpha_4\beta_3^2\beta_4^2 - \alpha_1^4\alpha_4\beta_4^2 + \alpha_1^2\alpha_3^2\beta_3\beta_4^2 + \alpha_1^3\alpha_4\beta_3\beta_4^2 - \alpha_1^4\alpha_3^2\alpha_4 - 4\alpha_1^3\alpha_3^2\alpha_4\beta_3$$
$$-3\alpha_3\alpha_4\beta_3^4\beta_4 - 7\alpha_1\alpha_3\alpha_4\beta_3^3\beta_4 + \alpha_4^2\beta_1\beta_3^3\beta_4 - 3\alpha_1^2\alpha_3\alpha_4\beta_3^2\beta_4 + 3\alpha_1\alpha_4^2\beta_1\beta_3^2\beta_4$$
$$\left.+2\alpha_1^4\alpha_3\alpha_4\beta_4 + \alpha_1^3\alpha_4^2\beta_1\beta_4 + 3\alpha_1^3\alpha_3\alpha_4\beta_3\beta_4 + 3\alpha_1^2\alpha_4^2\beta_1\beta_3\beta_4\right) \Big/ \Big((\alpha_2 + \beta_4)$$
$$\big(2\alpha_1^2\beta_4^2 + 3\alpha_4\alpha_1^2\beta_3 + 3\alpha_2\alpha_1^2\beta_4 - 2\alpha_3\alpha_1^2\beta_4 + 3\alpha_4\alpha_1\beta_3^2 + \alpha_1\beta_3\beta_4^2 - \alpha_2^2\alpha_1\beta_3$$
$$-4\alpha_2\alpha_3\alpha_1\beta_3 - 4\alpha_3\alpha_1\beta_3\beta_4 + \alpha_4\beta_3^3 - 2\alpha_2^2\beta_3^2 - 2\alpha_2\alpha_3\beta_3^2 + \alpha_2^3\beta_1 + 3\alpha_2\beta_1\beta_4^2$$
$$-3\alpha_2\beta_3^2\beta_4 - 2\alpha_3\beta_3^2\beta_4 + 3\alpha_2^2\beta_1\beta_4 + \alpha_4\alpha_1^3 + \alpha_2^2\alpha_1^2 - 2\alpha_2\alpha_3\alpha_1^2$$
$$+\beta_1\beta_4^3 - \beta_3^2\beta_4^2\big)^{\frac{3}{2}}\Big);$$

and for such algebras the system of functions $\lambda,\ \alpha_{1,c},\ \beta_{1,c}^2$ *are the separating system of invariants.*

- $\alpha_{4,2}(A) = 0$
  - $*$ $$\alpha_{2,2}(A) = -\alpha_1^2\beta_4 + \alpha_2\alpha_1\beta_2 + \alpha_2\alpha_1\beta_3 + \alpha_1\beta_2\beta_4 - \alpha_1\beta_3\beta_4 + \alpha_2\beta_3^2 - \alpha_2^2\beta_1 + \alpha_2\beta_2\beta_3 - 2\alpha_2\beta_1\beta_4 - \beta_1\beta_4^2 + \beta_2\beta_3\beta_4 \neq 0.$$

    Respectively,
    $$\alpha'_{2,2}(A) = -\alpha_1\beta_4^2 - 2\alpha_4\alpha_1\beta_3 - \alpha_2\alpha_1\beta_4 + \alpha_3\alpha_1\beta_4 - \alpha_4\beta_3^2 + \alpha_2^2\beta_3 + \alpha_2\alpha_3\beta_3 + \alpha_2\beta_3\beta_4 + \alpha_3\beta_3\beta_4 - \alpha_4\alpha_1^2 + \alpha_2\alpha_3\alpha_1 \neq 0.$$

    then
    $$A \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_{1,c} & \lambda & 1 & -1 \end{pmatrix} = A_3,$$

    where
    $$\beta_{1,c} = -\frac{\big(\alpha_1\beta_3+\alpha_2\beta_1+\alpha_1^2+\beta_1\beta_4\big)\big(2\alpha_1\beta_2+3\alpha_1\beta_3-3\alpha_2\beta_1+\alpha_1^2+2\beta_3^2+2\beta_2\beta_3-3\beta_1\beta_4\big)}{4(\alpha_1+\beta_3)^4} - \frac{\beta_1\big(\alpha_1^2\beta_4-\alpha_2\alpha_1\beta_2-\alpha_2\alpha_1\beta_3-\alpha_1\beta_2\beta_4+\alpha_1\beta_3\beta_4-\alpha_2\beta_3^2+\alpha_2^2\beta_1\big)}{(\alpha_1+\beta_3)^4} + \frac{\beta_1\big(\alpha_2\beta_2\beta_3-2\alpha_2\beta_1\beta_4-\beta_1\beta_4^2+\beta_2\beta_3\beta_4\big)}{(\alpha_1+\beta_3)^4}.$$

    Respectively,
    $$\beta_{1,c} = -\frac{\big(\alpha_2\beta_4+\alpha_4\beta_3+\alpha_1\alpha_4+\beta_4^2\big)\big(3\alpha_2\beta_4-3\alpha_4\beta_3+2\alpha_3\beta_4+2\alpha_2^2+2\alpha_3\alpha_2-3\alpha_1\alpha_4+\beta_4^2\big)}{4(\alpha_2+\beta_4)^4} - \frac{\alpha_4\big(\alpha_1\beta_4^2+2\alpha_4\alpha_1\beta_3+\alpha_2\alpha_1\beta_4-\alpha_3\alpha_1\beta_4+\alpha_4\beta_3^2-\alpha_2^2\beta_3-\alpha_2\alpha_3\beta_3-\alpha_2\beta_3\beta_4\big)}{(\alpha_2+\beta_4)^4} + \frac{\alpha_4\big(\alpha_3\beta_3\beta_4-\alpha_4\alpha_1^2+\alpha_1\alpha_2\alpha_3\big)}{(\alpha_2+\beta_4)^4};$$

    and for such algebras the system of functions $\lambda,\ \beta_{1,c}$ *are the separating system of invariants.*
  - $*$ $\alpha_{2,2}(A) = 0$,
    - $\star$ $\beta_{1,2}(A) = 1 + \frac{\alpha_1+\beta_2}{\alpha_1+\beta_3} - 3(\frac{\alpha_1}{\alpha_1+\beta_3} + \beta_1\frac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2}) \neq 0$ *or* $\beta_{1,2}(A) = 0,\ \beta_1 = 0.$
      Respectively, $\alpha'_{2,2}(A) = 0$, $\beta'_{1,2}(A) = 1 + \frac{\alpha_3+\beta_4}{\alpha_2+\beta_4} - 3(\frac{\beta_4}{\alpha_2+\beta_4} + \alpha_4\frac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2}) \neq 0$ *or* $\beta'_{1,2}(A) = 0,\ \alpha_4 = 0\big)$ *then*
      $$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 0 \\ 0 & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 0 \end{pmatrix} = A_4,$$
      where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_3} + \beta_1\frac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2}$.
      Respectively, $\alpha_{1,c} = \frac{\beta_4}{(\alpha_2+\beta_4)} + \alpha_4\frac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2}$.
    - $\star$ $\beta_{1,2}(A) = 0$, $\beta_1 \neq 0$.
      Respectively, $\alpha'_{2,2}(A) = 0$, $\beta'_{1,2}(A) = 0$ *and* $\alpha_4 \neq 0$, *then*
      $$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 0 \\ 1 & 2\alpha_{1,c}-1 & 1-\alpha_{1,c} & 0 \end{pmatrix} = A_5,$$

where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_3} + \beta_1 \frac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2}$.
Respectively, $\alpha_{1,c} = \frac{\beta_4}{\alpha_2+\beta_4} + \alpha_4 \frac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2}$).

C. *If* $Tr_1(A) = (0,0) = (\alpha_1+\beta_3, \alpha_2+\beta_4)$,

- $\alpha_1+\beta_2 \neq 0$ ($\alpha_3+\beta_4 \neq 0$)
  - $\alpha_{4,3}(A) = -\frac{\beta_1(\alpha_2-\alpha_3)^3}{(\alpha_1+\beta_2)^2} + \frac{3\alpha_1(\alpha_2-\alpha_3)^2}{\alpha_1+\beta_2} + \alpha_4\beta_2 - 2\alpha_3^2 + \alpha_2^2 + \alpha_2\alpha_3 + \alpha_1\alpha_4 \neq 0$.
    Respectively, $\alpha'_{4,3}(A) = -\frac{\alpha_4(\beta_3-\beta_2)^3}{(\alpha_3+\beta_4)^2} + \frac{3\beta_4(\beta_3-\beta_2)^2}{\alpha_3+\beta_4} + \alpha_3\beta_1 - 2\beta_2^2 + \beta_3^2 + \beta_2\beta_3 + \beta_1\beta_4 \neq 0$, *then*

$$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 1 \\ \beta_{1,c} & 1-\alpha_{1,c} & -\alpha_{1,c} & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 1 \\ -\beta_{1,c} & 1-\alpha_{1,c} & -\alpha_{1,c} & 0 \end{pmatrix} = A_6,$$

where

$$\begin{aligned}
\alpha_{1,c} \;=\;& (2\alpha_4\alpha_1^2\beta_2 + \alpha_4\alpha_1\beta_2^2 - \alpha_2\alpha_4\alpha_1\beta_1 + \alpha_3\alpha_4\alpha_1\beta_1 + \alpha_2^2\alpha_1\beta_2 - 3\alpha_2\alpha_3\alpha_1\beta_2 \\
& -\alpha_3^2\beta_2^2 - \alpha_2^3\beta_1 - \alpha_2\alpha_3^2\beta_1 + 2\alpha_2^2\alpha_3\beta_1 - \alpha_2\alpha_4\beta_1\beta_2 + \alpha_3\alpha_4\beta_1\beta_2 + \alpha_4\alpha_1^3 \\
& -\alpha_1^2\alpha_2\alpha_3)\big/(3\alpha_4\alpha_1^2\beta_2 + 3\alpha_4\alpha_1\beta_2^2 + 5\alpha_2^2\alpha_1\beta_2 - \alpha_3^2\alpha_1\beta_2 - 4\alpha_2\alpha_3\alpha_1\beta_2 \\
& +\alpha_4\beta_2^3 + \alpha_2^2\beta_2^2 - 2\alpha_3^2\beta_2^2 + \alpha_2\alpha_3\beta_2^2 - \alpha_2^3\beta_1 + \alpha_3^3\beta_1 - 3\alpha_2\alpha_3^2\beta_1 + 3\alpha_2^2\alpha_3\beta_1 \\
& +\alpha_4\alpha_1^3 + 4\alpha_2^2\alpha_1^2 + \alpha_3^2\alpha_1^2 - 5\alpha_2\alpha_3\alpha_1^2), \\
\beta_{1,c} \;=\;& (\alpha_4^2\alpha_1^3\beta_1 + 6\alpha_2\alpha_4\alpha_1^3\beta_2 - \alpha_3\alpha_4\alpha_1^3\beta_2 + 3\alpha_3\alpha_4\alpha_1^2\beta_2^2 - 3\alpha_3^2\alpha_4\alpha_1^2\beta_1 \\
& +3\alpha_2\alpha_3\alpha_4\alpha_1^2\beta_1 - 4\alpha_2^3\alpha_1^2\beta_2 + \alpha_3^3\alpha_1^2\beta_2 - 3\alpha_2\alpha_3^2\alpha_1^2\beta_2 + 3\alpha_4^2\alpha_1^2\beta_1\beta_2 \\
& -2\alpha_2\alpha_4\alpha_1\beta_2^3 + \alpha_3\alpha_4\alpha_1\beta_2^3 - 2\alpha_2^3\alpha_1\beta_2^2 - \alpha_3^3\alpha_1\beta_2^2 - 3\alpha_2^2\alpha_3\alpha_1\beta_2^2 \\
& +3\alpha_4^2\alpha_1\beta_1\beta_2^2 + 4\alpha_2^4\alpha_1\beta_1 + 2\alpha_2\alpha_3^3\alpha_1\beta_1 - 6\alpha_2^3\alpha_3\alpha_1\beta_1 + 3\alpha_2^2\alpha_4\alpha_1\beta_1\beta_2 \\
& -3\alpha_3^2\alpha_4\alpha_1\beta_1\beta_2 - \alpha_3\alpha_4\beta_2^4 - \alpha_2\alpha_3^2\beta_2^3 - \alpha_2^2\alpha_3\beta_2^3 + \alpha_4^2\beta_1\beta_2^3 + \alpha_2^3\alpha_4\beta_1^2 \\
& -\alpha_3^3\alpha_4\beta_1^2 + 3\alpha_2\alpha_3^2\alpha_4\beta_1^2 - 3\alpha_2^2\alpha_3\alpha_4\beta_1^2 + 3\alpha_2^2\alpha_4\beta_1\beta_2^2 - 3\alpha_2\alpha_3\alpha_4\beta_1\beta_2^2 \\
& +2\alpha_2^4\beta_1\beta_2 + \alpha_3^4\beta_1\beta_2 + \alpha_2\alpha_3^3\beta_1\beta_2 - 3\alpha_2^2\alpha_3^2\beta_1\beta_2 - \alpha_2^3\alpha_3\beta_1\beta_2 + 4\alpha_2\alpha_4\alpha_1^4 \\
& -2\alpha_3\alpha_4\alpha_1^4 + 2\alpha_2\alpha_3^2\alpha_1^3 - 4\alpha_2^2\alpha_3\alpha_1^3)\big/\big(3\alpha_4\alpha_1^2\beta_2 + 3\alpha_4\alpha_1\beta_2^2 + 5\alpha_2^2\alpha_1\beta_2 \\
& -\alpha_3^2\alpha_1\beta_2 - 4\alpha_2\alpha_3\alpha_1\beta_2 + \alpha_4\beta_2^3 + \alpha_2^2\beta_2^2 - 2\alpha_3^2\beta_2^2 + \alpha_2\alpha_3\beta_2^2 - \alpha_2^3\beta_1 \\
& +\alpha_3^3\beta_1 - 3\alpha_2\alpha_3^2\beta_1 + 3\alpha_2^2\alpha_3\beta_1 + \alpha_4\alpha_1^3 + 4\alpha_2^2\alpha_1^2 + \alpha_3^2\alpha_1^2 - 5\alpha_2\alpha_3\alpha_1^2\big)^{3/2}.
\end{aligned}$$

Respectively,

$$\begin{aligned}
\alpha_{1,c} \;=\;& \big(-\alpha_4\beta_3^3 + 2\alpha_4\beta_2\beta_3^2 + \alpha_3\beta_4\beta_3^2 - \alpha_4\beta_2^2\beta_3 - \alpha_3\alpha_4\beta_1\beta_3 - \alpha_4\beta_1\beta_4\beta_3 \\
& -3\alpha_3\beta_2\beta_4\beta_3 - \alpha_3^2\beta_2^2 + 2\alpha_3\beta_1\beta_4^2 + \alpha_3\alpha_4\beta_1\beta_2 + \alpha_3^2\beta_1\beta_4 + \alpha_4\beta_1\beta_2\beta_4 \\
& -\beta_2\beta_4^2\beta_3 + \beta_1\beta_4^3\big)\big/\big(\alpha_3^3\beta_1 - 2\alpha_3^2\beta_2^2 + \alpha_3^2\beta_3^2 + \alpha_3^2\beta_2\beta_3 + 3\alpha_3^2\beta_1\beta_4 \\
& +3\alpha_3\beta_1\beta_4^2 - \alpha_3\beta_2^2\beta_4 + 5\alpha_3\beta_3^2\beta_4 - 4\alpha_3\beta_2\beta_3\beta_4 + \alpha_4\beta_2^3 - \alpha_4\beta_3^3 + 3\alpha_4\beta_2\beta_3^2 \\
& -3\alpha_4\beta_2^2\beta_3 + \beta_1\beta_4^3 + \beta_2^2\beta_4^2 + 3\alpha_3\beta_1\beta_4^2 + 4\beta_3^2\beta_4^2 - 5\beta_2\beta_3\beta_4^2\big), \\
\beta_{1,c} \;=\;& \Big(-\alpha_3^4\beta_1\beta_2 + \alpha_4\alpha_3^3\beta_1^2 - \alpha_3^3\beta_2\beta_3^2 - \alpha_3^3\beta_2^2\beta_3 + \alpha_3^3\beta_1\beta_2\beta_4 - 2\alpha_3^3\beta_1\beta_3\beta_4 \\
& +3\alpha_4\alpha_3^2\beta_1\beta_3^2 + 3\alpha_3^2\beta_1\beta_2\beta_4^2 - 3\alpha_4\alpha_3^2\beta_1\beta_2\beta_3 - \alpha_3^2\beta_2^3\beta_4 - 2\alpha_3^2\beta_3^3\beta_4 \\
& +3\alpha_4\alpha_3^2\beta_1^2\beta_4 - 3\alpha_3^2\beta_2\beta_3^2\beta_4 + \alpha_4\alpha_3\beta_2^4 + 2\alpha_4\alpha_3\beta_3^4 - \alpha_4\alpha_3\beta_2\beta_3^3 - \alpha_3\beta_1\beta_2\beta_4^3 \\
& +3\alpha_3\beta_3(2\beta_1\beta_4^3 - \alpha_4\beta_2^2\beta_3) + \alpha_3\beta_4^2(\beta_2^3 - 4\beta_3^3) + 3\alpha_3\beta_4^2(\alpha_4\beta_1^2 - \beta_2^2\beta_3) \\
& +\alpha_4\alpha_3\beta_2^3\beta_3 - 3\alpha_3\alpha_4\beta_1\beta_4(\beta_2^2 - -\beta_3^2) - \alpha_4^2\beta_1(\beta_2^3 - \beta_3^3) + \alpha_4\beta_1^2\beta_4^3 \\
& -3\alpha_4^2\beta_1\beta_2\beta_3^2 - 3\alpha_4\beta_1\beta_2^2\beta_4^2 + 3\alpha_4\beta_1\beta_2\beta_3\beta_4^2 + 3\alpha_4^2\beta_1\beta_2^2\beta_3 + 4\alpha_4\beta_3^4\beta_4 \\
& -6\alpha_4\beta_2\beta_3^3\beta_4 + 2\alpha_4\beta_2^3\beta_3\beta_4 - 2\beta_1\beta_2\beta_4^4 + 4\beta_1\beta_3\beta_4^4 - 4\beta_2\beta_3^2\beta_4^3 + 2\beta_2^2\beta_3\beta_4^3\big) \\
& [\alpha_3^2(\alpha_3\beta_1 - 2\beta_2^2 + \beta_3^2 + \beta_2\beta_3) + 3\alpha_3\beta_1\beta_4(\alpha_3+\beta_4) - \alpha_3\beta_4(\beta_2^2 - 5\beta_3^2) \\
& -4\alpha_3\beta_2\beta_3\beta_4 + \alpha_4\beta_2^3 - \alpha_4\beta_3^3 + 3\alpha_4\beta_2\beta_3^2 - 3\alpha_4\beta_2^2\beta_3 + \beta_1\beta_4^3 + \beta_2^2\beta_4^2 \\
& +4\beta_3^2\beta_4^2 - 5\beta_2\beta_3\beta_4^2]^{3/2}\Big);
\end{aligned}$$

with the separating system $\alpha_{1,c},\ \beta_{1,c}^2$.

- $\alpha_{4,3}(A) = 0$,

* $\alpha_{2,3}(A) = \frac{2\alpha_2\alpha_1\beta_2+\alpha_3\beta_2^2-\alpha_2^2\beta_1-\alpha_3^2\beta_1+2\alpha_2\alpha_3\beta_1+2\alpha_2\alpha_1^2-\alpha_3\alpha_1^2}{(\alpha_1+\beta_2)^2} \neq 0.$
Respectively, $\alpha'_{4,3}(A) = 0$,
$\alpha'_{2,3}(A) = \frac{1}{(\alpha_3+\beta_4)^2}\left(\alpha_3^2\beta_2 + 2\alpha_3\beta_3\beta_4 - \alpha_4\beta_2^2 - \alpha_4\beta_3^2 + 2\alpha_4\beta_2\beta_3 - \beta_2\beta_4^2 + 2\beta_3\beta_4^2) \neq 0\right)$,
then
$$A \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_{1,c} & 1 & 0 & -1 \end{pmatrix} = A_7,$$
where
$\beta_{1,c} = \frac{-\alpha_1^2(3\beta_2^2-4\alpha_2\beta_1)-2\alpha_1\beta_2(\beta_2^2-3\alpha_2\beta_1)+2\alpha_3\beta_1\beta_2(\alpha_1+\beta_2)-\beta_1^2(\alpha_2^2+\alpha_3^2)+2\alpha_2\beta_1(\alpha_3\beta_1+\beta_2^2)+\alpha_1^4}{4(\alpha_1+\beta_2)^4}.$
Respectively,
$\beta_{1,c} = \frac{-\alpha_3^2\beta_4(3\beta_4+2\alpha_3)+2\alpha_4\beta_3(2\beta_4^2+\alpha_4\beta_2)+2\alpha_3\alpha_4\beta_4(\beta_2+3\beta_3)-\alpha_4^2(\beta_2^2+\beta_3^2)+2\alpha_3^2\alpha_4(\beta_2+\beta_3)+\beta_4^4}{4(\alpha_3+\beta_4)^4}),$
and $\beta_{1,c}$ *separates such non-isomorphic algebras.*

* $\alpha_{2,3}(A) = 0$,

  ⋆ $1 - 3(\frac{\alpha_1}{\alpha_1+\beta_2} + \beta_1\frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2}) \neq 0$ *or* $(1 - 3(\frac{\alpha_1}{\alpha_1+\beta_2} + \beta_1\frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2}) = 0,$ *and* $\beta_1 = 0.$
  Respectively, $\alpha'_{4,3}(A) = 0$, $\alpha'_{2,3}(A) = 0$,
  $1 - 3(\frac{\beta_4}{\alpha_3+\beta_4} + \alpha_4\frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2}) \neq 0$ *or* $1 - 3(\frac{\beta_4}{\alpha_3+\beta_4} + \alpha_4\frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2}) = 0,\ \alpha_4 = 0$, *then*
$$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 0 \\ 0 & 1-\alpha_{1,c} & -\alpha_{1,c} & 0 \end{pmatrix} = A_8,$$
  where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_2} + \beta_1\frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2}.$
  Respectively, $\alpha_{1,c} = \frac{\beta_4}{\alpha_3+\beta_4} + \alpha_4\frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2}.$

  ⋆ $1 - 3(\frac{\alpha_1}{\alpha_1+\beta_2} + \beta_1\frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2}) = 0$, $\beta_1 \neq 0$.
  Respectively, $\alpha'_{4,3}(A) = 0$, $\alpha'_{2,3}(A) = 0$, $1 - 3(\frac{\beta_4}{\alpha_3+\beta_4} + \alpha_4\frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2}) = 0$, $\alpha_4 \neq 0$, *then*
$$A \simeq \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix} = A_9.$$

D. *If* $Tr_1(A) = Tr_2(A) = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_4 = 0, \beta_1 = 0$, *then*
$$A \simeq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is the trivial algebra}.$$

E. *If* $Tr_1(A) = Tr_2(A) = 0$, $\alpha_2^2 = \alpha_1\alpha_4$, $\alpha_2^3 = -\beta_1\alpha_4^2$ *and* $\alpha_4 \neq 0$ *or* $\alpha_4 = \alpha_2 = \alpha_1 = 0$ *and* $\beta_1 \neq 0$, *then*
$$A \simeq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = A_{12}.$$

F. *If* $Tr_1(A) = Tr_2(A) = 0$ *and* $\alpha_4 = 0$, $\alpha_2 \neq 0$, $3\alpha_1^2 + 4\alpha_2\beta_1 \neq 0$, *then*
$$A \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} = A_{11}.$$

G. *In all other cases of* $Tr_1(A) = Tr_2(A) = 0$,
$$A \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = A_{10}.$$

Proof. We divide $Mat(2\times 4,\mathbb{F})$ into the following four disjoint invariant subsets:

1. $\mathfrak{S}_1=\{A\in Mat(2\times 4,\mathbb{F})|\ \{Tr_1(A),Tr_2(A)\}$ is linear independent$\}$;
2. $\mathfrak{S}_2=\{A\in Mat(2\times 4,\mathbb{F})|\ Tr_1(A)\neq 0$ and $Tr_2(A)=\lambda Tr_1(A),$ where $\lambda\in\mathbb{F}\}$;
3. $\mathfrak{S}_3=\{A\in Mat(2\times 4,\mathbb{F})|\ Tr_1(A)=0$ and $Tr_2(A)\neq 0\}$;
4. $\mathfrak{S}_4=\{A\in Mat(2\times 4,\mathbb{F})|\ Tr_1(A)=Tr_2(A)=(0,0)\}$.

Consider $\mathfrak{S}_1$. In this case as $P(A)$ one can take the following nonsingular matrix

$$P(A)=\begin{pmatrix}\alpha_1+\beta_3 & \alpha_2+\beta_4\\ \alpha_1+\beta_2 & \alpha_3+\beta_4\end{pmatrix}.$$

Then the equality $P(gA(g^{-1})^{\otimes 2})=P(A)g^{-1}$ is valid at any $g\in GL(2,\mathbb{F})$, and due to Lemma 2.2 for the canonical MSC of A we have

$$A_1=P(A)A(P(A)^{-1}\otimes P(A)^{-1})=\begin{pmatrix}\alpha_{1,c} & \alpha_{2,c} & \alpha_{2,c}+1 & \alpha_{4,c}\\ \beta_{1,c} & -\alpha_{1,c} & 1-\alpha_{1,c} & -\alpha_{2,c}\end{pmatrix},$$

where

$$\begin{aligned}
\alpha_{1,c} \;=\;& \tfrac{1}{\Delta^2}(-\alpha_1^2\alpha_2\alpha_3+\alpha_1^3\alpha_4+\alpha_2\alpha_3^2\beta_1-2\alpha_1\alpha_2\alpha_3\beta_2-\alpha_1\alpha_3^2\beta_2+2\alpha_1^2\alpha_4\beta_2-\alpha_2\alpha_3\beta_2^2+\alpha_1\alpha_4\beta_2^2\\
&-2\alpha_1\alpha_2\alpha_3\beta_3+\alpha_1^2\alpha_4\beta_3-2\alpha_2\alpha_3\beta_2\beta_3-\alpha_3^2\beta_2\beta_3+2\alpha_1\alpha_4\beta_2\beta_3+\alpha_4\beta_2^2\beta_3+\alpha_1^2\alpha_3\beta_4\\
&+2\alpha_2\alpha_3\beta_1\beta_4+\alpha_3^2\beta_1\beta_4-2\alpha_1\alpha_3\beta_2\beta_4-\alpha_3\beta_2^2\beta_4-2\alpha_1\alpha_2\beta_3\beta_4-2\alpha_2\beta_2\beta_3\beta_4-2\alpha_3\beta_2\beta_3\beta_4\\
&+2\alpha_1^2\beta_4^2+\alpha_2\beta_1\beta_4^2+2\alpha_3\beta_1\beta_4^2+\alpha_1\beta_2\beta_4^2-\beta_2\beta_3\beta_4^2+\beta_1\beta_4^3)\\
\alpha_{2,c} \;=\;& \tfrac{1}{\Delta^2}(\alpha_1^2\alpha_2\alpha_3-\alpha_1^3\alpha_4-\alpha_2^2\alpha_3\beta_1+2\alpha_1\alpha_2\alpha_3(\beta_2+\beta_3)-\alpha_1^2\alpha_4\beta_2+\alpha_1\alpha_2^2\beta_3-2\alpha_1^2\alpha_4\beta_3\\
&+\alpha_2^2\beta_2\beta_3+2\alpha_2\alpha_3\beta_2\beta_3-2\alpha_1\alpha_4\beta_2\beta_3+\alpha_2\alpha_3\beta_3^2-\alpha_1\alpha_4\beta_3^2-\alpha_4\beta_2\beta_3^2-\alpha_1^2\alpha_2\beta_4-\alpha_2^2\beta_1\beta_4\\
&-2\alpha_2\alpha_3\beta_1\beta_4+2\alpha_1\alpha_3\beta_2\beta_4+2\alpha_1\alpha_2\beta_3\beta_4+2\alpha_2\beta_2\beta_3\beta_4+2\alpha_3\beta_2\beta_3\beta_4+\alpha_2\beta_3^2\beta_4-2\alpha_1^2\beta_4^2\\
&-2\alpha_2\beta_1\beta_4^2-\alpha_3\beta_1\beta_4^2-\alpha_1\beta_3\beta_4^2+\beta_2\beta_3\beta_4^2-\beta_1\beta_4^3)\\
\alpha_{4,c} \;=\;& \tfrac{1}{\Delta^2}(-\alpha_1^2\alpha_2\alpha_3+\alpha_1^3\alpha_4+\alpha_2^3\beta_1-\alpha_1\alpha_2^2\beta_2-2\alpha_1\alpha_2^2\beta_3-2\alpha_1\alpha_2\alpha_3\beta_3+3\alpha_1^2\alpha_4\beta_3-\alpha_2^2\beta_2\beta_3\\
&-2\alpha_2^2\beta_3^2-\alpha_2\alpha_3\beta_3^2+3\alpha_1\alpha_4\beta_3^2+\alpha_4\beta_3^3+2\alpha_1^2\alpha_2\beta_4-\alpha_1^2\alpha_3\beta_4+3\alpha_2^2\beta_1\beta_4-2\alpha_1\alpha_2\beta_2\beta_4\\
&-2\beta_3\beta_4(\alpha_1\alpha_3+\alpha_2\beta_2+\alpha_2\beta_3)-\alpha_3\beta_3^2\beta_4+2\alpha_1^2\beta_4^2+3\alpha_2\beta_1\beta_4^2-\alpha_1\beta_2\beta_4^2+2\alpha_1\beta_3\beta_4^2\\
&-\beta_2\beta_3\beta_4^2+\beta_1\beta_4^3)\\
\beta_{1,c} \;=\;& \tfrac{1}{\Delta^2}(-\alpha_1^2\alpha_2\alpha_3+\alpha_1^3\alpha_4+\alpha_3^3\beta_1-2\alpha_1\alpha_2\alpha_3\beta_2-2\alpha_1\alpha_3^2\beta_2+3\alpha_1^2\alpha_4\beta_2-\alpha_2\alpha_3\beta_2^2-2\alpha_3^2\beta_2^2\\
&+3\alpha_1\alpha_4\beta_2^2+\alpha_4\beta_2^3-\alpha_1\alpha_3^2\beta_3-\alpha_3^2\beta_2\beta_3-\alpha_1^2\alpha_2\beta_4+2\alpha_1^2\alpha_3\beta_4+3\alpha_3^2\beta_1\beta_4-2\alpha_1\alpha_2\beta_2\beta_4\\
&-\alpha_2\beta_2^2\beta_4-2\alpha_3\beta_4(\beta_2^2+\alpha_1\beta_3+\beta_2\beta_3)+2\alpha_1^2\beta_4^2+3\alpha_3\beta_1\beta_4^2+2\alpha_1\beta_2\beta_4^2-\alpha_1\beta_3\beta_4^2\\
&-\beta_2\beta_3\beta_4^2+\beta_1\beta_4^3)
\end{aligned}$$

and $\Delta=\Delta(A)=\det(P(A))=\alpha_1\alpha_3-\alpha_1\alpha_2-\alpha_2\beta_2+\alpha_3\beta_3-\beta_2\beta_4+\beta_3\beta_4\neq 0.$

These four invariant rational functions $\alpha_{1,c}$, $\alpha_{2,c}$, $\alpha_{4,c}$ and $\beta_{1,c}$ separate non-isomorphic algebras from $\mathfrak{S}_1$.

Consider $\mathfrak{S}_2$. Note that $B=\begin{pmatrix}\beta_4 & \beta_3 & \beta_2 & \beta_1\\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1\end{pmatrix}\simeq A$ and

$$Tr_1(B)=(\alpha_2+\beta_4,\alpha_1+\beta_3),\ \ Tr_2(B)=(\alpha_3+\beta_4,\alpha_1+\beta_2).$$

In fact, $B=gA(g^{-1})^{\otimes 2}$ for $g=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ therefore, in this case we consider only $\alpha_1+\beta_3\neq 0$ case in details as far as due to the above if $\alpha_2+\beta_4\neq 0$ and the corresponding result can be directly derived from $\alpha_1+\beta_3\neq 0$ case. They are given in parentheses at the appropriate places.

If $\alpha_1+\beta_3\neq 0$ $(\alpha_2+\beta_4\neq 0)$ put $\lambda=\frac{\alpha_1+\beta_2}{\alpha_1+\beta_3}$ $\left(\text{respectively, } \lambda=\frac{\alpha_3+\beta_4}{\alpha_2+\beta_4}\right)$ and at $g=\begin{pmatrix}\alpha_1+\beta_3 & \alpha_2+\beta_4\\ 0 & 1\end{pmatrix}$ $\left(\text{respectively, } g=\begin{pmatrix}\alpha_1+\beta_3 & \alpha_2+\beta_4\\ 1 & 0\end{pmatrix}\right)$ one has

$$A'=gA(g^{-1})^{\otimes 2}=\begin{pmatrix}\alpha_1' & \alpha_2' & \alpha_2' & \alpha_4'\\ \beta_1' & \lambda-\alpha_1' & 1-\alpha_1' & -\alpha_2'\end{pmatrix}, \text{ where}$$

$$\begin{aligned}
\alpha_1' &= \tfrac{\alpha_1}{\alpha_1+\beta_3}+\beta_1\tfrac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2},\\
\alpha_2' &= \tfrac{1}{(\alpha_1+\beta_3)^2}\big(-\alpha_1^2\beta_4+\alpha_2\alpha_1\beta_2+\alpha_2\alpha_1\beta_3+\alpha_1\beta_2\beta_4-\alpha_1\beta_3\beta_4+\alpha_2\beta_3^2-\alpha_2^2\beta_1+\alpha_2\beta_2\beta_3\\
&\quad -2\alpha_2\beta_1\beta_4-\beta_1\beta_4^2+\beta_2\beta_3\beta_4\big)\\
\alpha_4' &= \tfrac{1}{(\alpha_1+\beta_3)^2}\big(-2\alpha_1^2\beta_4^2\lambda-4\alpha_2\alpha_1^2\beta_4\lambda-4\alpha_1\beta_3\beta_4^2\lambda-4\alpha_2^2\alpha_1\beta_3\lambda-8\alpha_2\alpha_1\beta_3\beta_4\lambda-2\alpha_2^2\beta_3^2\lambda\\
&\quad -4\alpha_2\beta_3^2\beta_4\lambda+4\alpha_1^2\beta_4^2+3\alpha_4\alpha_1^2\beta_3+5\alpha_2\alpha_1^2\beta_4+3\alpha_4\alpha_1\beta_3^2+5\alpha_1\beta_3\beta_4^2-\alpha_2^2\alpha_1\beta_3\\
&\quad +4\alpha_2\alpha_1\beta_3\beta_4+\alpha_4\beta_3^3-2\alpha_2^2\beta_3^2+3\alpha_2\beta_1\beta_4^2+\alpha_2^3\beta_1-\alpha_2\beta_3^2\beta_4+3\alpha_2^2\beta_1\beta_4-2\alpha_2^2\alpha_1^2\lambda\\
&\quad +\alpha_4\alpha_1^3+\alpha_2^2\alpha_1^2-2\beta_3^2\beta_4^2\lambda+\beta_1\beta_4^3+\beta_3^2\beta_4^2\big)\\
\beta_1' &= \tfrac{\beta_1}{(\alpha_1+\beta_3)^2}
\end{aligned}$$

(respectively,

$$\begin{aligned}
\alpha_1' &= \tfrac{\beta_4}{\alpha_2+\beta_4}+\alpha_4\tfrac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2},\\
\alpha_2' &= \tfrac{1}{(\alpha_2+\beta_4)^2}\big(-\alpha_1\beta_4^2-2\alpha_4\alpha_1\beta_3-\alpha_2\alpha_1\beta_4+\alpha_3\alpha_1\beta_4-\alpha_4\beta_3^2+\alpha_2^2\beta_3+\alpha_2\alpha_3\beta_3+\alpha_2\beta_3\beta_4\\
&\quad +\alpha_3\beta_3\beta_4-\alpha_4\alpha_1^2+\alpha_2\alpha_3\alpha_1\big)\\
\alpha_4' &= \tfrac{1}{(\alpha_2+\beta_4)^2}\big(-2\alpha_1^2\beta_4^2\lambda-4\alpha_2\alpha_1^2\beta_4\lambda-4\alpha_1\beta_3\beta_4^2\lambda-4\alpha_2^2\alpha_1\beta_3\lambda-8\alpha_2\alpha_1\beta_3\beta_4\lambda-2\alpha_2^2\beta_3^2\lambda\\
&\quad -4\alpha_2\beta_3^2\beta_4\lambda+4\alpha_1^2\beta_4^2+3\alpha_4\alpha_1^2\beta_3+5\alpha_2\alpha_1^2\beta_4+3\alpha_4\alpha_1\beta_3^2+5\alpha_1\beta_3\beta_4^2-\alpha_2^2\alpha_1\beta_3+4\alpha_2\alpha_1\beta_3\beta_4\\
&\quad +4\alpha_2\alpha_1\beta_3\beta_4+\alpha_4\beta_3^3-2\alpha_2^2\beta_3^2+3\alpha_2\beta_1\beta_4^2+\alpha_2^3\beta_1-\alpha_2\beta_3^2\beta_4+3\alpha_2^2\beta_1\beta_4-2\alpha_2^2\alpha_1^2\lambda+\alpha_4\alpha_1^3\\
&\quad +\alpha_2^2\alpha_1^2-2\beta_3^2\beta_4^2\lambda+\beta_1\beta_4^3+\beta_3^2\beta_4^2\big)\\
\beta_1' &= \tfrac{\alpha_4}{(\alpha_2+\beta_4)^2}\big)
\end{aligned}$$

Note that $Tr_1(A')=(1,0), Tr_2(A')=(\lambda,0)$ and only for matrices of the form $g^{-1}=\begin{pmatrix}1 & 0\\ \xi_2 & \eta_2\end{pmatrix}$ we have $(1,0)g^{-1}=(1,0)$ and

$$A''=gA'(g^{-1})^{\otimes 2}=\begin{pmatrix}\alpha_1'' & \alpha_2'' & \alpha_2'' & \alpha_4''\\ \beta_1'' & \lambda-\alpha_1'' & 1-\alpha_1'' & -\alpha_2''\end{pmatrix}, \text{ where}$$

$$\begin{aligned}
\alpha_1'' &= \alpha_1'+2\alpha_2'\xi_2+\alpha_4'\xi_2^2,\\
\alpha_2'' &= (\alpha_2'+\alpha_4'\xi_2)\eta_2,\\
\alpha_4'' &= \alpha_4'\eta_2^2,\\
\beta_1'' &= \frac{\beta_1'+\big(1+\lambda-3\alpha_1'\big)\xi_2-3\alpha_2'\xi_2^2-\alpha_4'\xi_2^3}{\eta_2}.
\end{aligned} \tag{1}$$

If $\alpha_4'\neq 0$, that is

$$\begin{aligned}
\alpha_{4,2} &= -2\alpha_1^2\beta_4^2\lambda-4\alpha_2\alpha_1^2\beta_4\lambda-4\alpha_1\beta_3\beta_4^2\lambda-4\alpha_2^2\alpha_1\beta_3\lambda-8\alpha_2\alpha_1\beta_3\beta_4\lambda-2\alpha_2^2\beta_3^2\lambda\\
&\quad -4\alpha_2\beta_3^2\beta_4\lambda+4\alpha_1^2\beta_4^2+3\alpha_4\alpha_1^2\beta_3+5\alpha_2\alpha_1^2\beta_4+3\alpha_4\alpha_1\beta_3^2+5\alpha_1\beta_3\beta_4^2-\alpha_2^2\alpha_1\beta_3\\
&\quad +4\alpha_2\alpha_1\beta_3\beta_4+\alpha_4\beta_3^3-2\alpha_2^2\beta_3^2+3\alpha_2\beta_1\beta_4^2+\alpha_2^3\beta_1-\alpha_2\beta_3^2\beta_4+3\alpha_2^2\beta_1\beta_4-2\alpha_2^2\alpha_1^2\lambda\\
&\quad +\alpha_4\alpha_1^3+\alpha_2^2\alpha_1^2-2\beta_3^2\beta_4^2\lambda+\beta_1\beta_4^3+\beta_3^2\beta_4^2\neq 0
\end{aligned}$$

then $\sqrt{\alpha_4''} = \sqrt{\alpha_4'}\eta_2,$, $\frac{\alpha_2''}{\sqrt{\alpha_4''}} = \left(\frac{\alpha_2'}{\sqrt{\alpha_4'}} + \sqrt{\alpha_4'}\xi_2\right)$, i.e., the equality

$$\begin{pmatrix} 1 & 0 \\ \frac{\alpha_2''}{\sqrt{\alpha_4''}} & \sqrt{\alpha_4''} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\alpha_2'}{\sqrt{\alpha_4'}} & \sqrt{\alpha_4'} \end{pmatrix} g^{-1}$$

holds true and therefore due to Lemma 2.2 for a canonical MSC one can take

$$\begin{aligned} A_2 = P(A')A'(P(A')^{-1})^{\otimes 2} &= \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 1 \\ \beta_{1,c} & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 0 \end{pmatrix} \\ &\simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 1 \\ -\beta_{1,c} & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 0 \end{pmatrix}, \end{aligned}$$

where $P(A') = \begin{pmatrix} 1 & 0 \\ \frac{\alpha_2'}{\sqrt{\alpha_4'}} & \sqrt{\alpha_4'} \end{pmatrix}$ and

$$\begin{aligned} \alpha_{1,c} = \; & \tfrac{1}{(\alpha_1+\beta_3)^2}\Big(\alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_1^2 + \beta_1\beta_4 - \big(\alpha_1^2\beta_4 - \alpha_2\alpha_1\beta_2 - \alpha_2\alpha_1\beta_3 - \alpha_1\beta_2\beta_4 \\ & +\alpha_1\beta_3\beta_4 - \alpha_2\beta_3^2 + \alpha_2^2\beta_1 - \alpha_2\beta_2\beta_3 + 2\alpha_2\beta_1\beta_4 + \beta_1\beta_4^2 - \beta_2\beta_3\beta_4\big)^2 \big/ \big(2\alpha_1^2\beta_4^2 \\ & +3\alpha_4\alpha_1^2\beta_3 + \alpha_2\alpha_1^2\beta_4 + 3\alpha_4\alpha_1\beta_3^2 - 2\alpha_1\beta_2\beta_4^2 + 3\alpha_1\beta_3\beta_4^2 - 2\alpha_2^2\alpha_1\beta_2 - 3\alpha_2^2\alpha_1\beta_3 \\ & -4\alpha_2\alpha_1\beta_2\beta_4 + \alpha_4\beta_3^3 - 2\alpha_2^2\beta_3^2 + 3\alpha_2\beta_1\beta_4^2 + \alpha_2^3\beta_1 - 2\alpha_2^2\beta_2\beta_3 - \alpha_2\beta_3^2\beta_4 + 3\alpha_2^2\beta_1\beta_4 \\ & -4\alpha_2\beta_2\beta_3\beta_4 + \alpha_4\alpha_1^3 - \alpha_1^2\alpha_2^2 + \beta_1\beta_4^3 + \beta_3^2\beta_4^2 - 2\beta_2\beta_3\beta_4^2\big)\Big) \end{aligned}$$

$$\begin{aligned} \beta_{1,c} = \; & \Big(\alpha_4\beta_4\alpha_1^5 + \alpha_2\beta_4^2\alpha_1^4 - \alpha_4^2\beta_1\alpha_1^4 - \alpha_2\alpha_4\beta_2\alpha_1^4 - \alpha_2\alpha_4\beta_3\alpha_1^4 - \alpha_2^2\beta_4\alpha_1^4 - 2\alpha_4\beta_2\beta_4\alpha_1^4 \\ & +2\alpha_4\beta_3\beta_4\alpha_1^4 - \beta_3\beta_4^3\alpha_1^3 + \alpha_2\alpha_4\beta_2^2\alpha_1^3 - 2\alpha_2\alpha_4\beta_3^2\alpha_1^3 - 2\alpha_2\beta_2\beta_4^2\alpha_1^3 + 3\alpha_2\beta_3\beta_4^2\alpha_1^3 \\ & +3\alpha_2^2\alpha_4\beta_1\alpha_1^3 + \alpha_2^3\beta_2\alpha_1^3 + \alpha_2^3\beta_3\alpha_1^3 - 4\alpha_4^2\beta_1\beta_3\alpha_1^3 - \alpha_2\alpha_4\beta_2\beta_3\alpha_1^3 + \alpha_4\beta_2^2\beta_4\alpha_1^3 \\ & +3\alpha_2\alpha_4\beta_1\beta_4\alpha_1^3 - \alpha_2^2\beta_2\beta_4\alpha_1^3 - 3\alpha_2^2\beta_3\beta_4\alpha_1^3 - \beta_1\beta_4^4 - 5\alpha_4\beta_2\beta_3\beta_4\alpha_1^3 - 2\beta_3^2\beta_4^3\alpha_1^2 \\ & +\alpha_2\beta_1\beta_4^3\alpha_1^2 + 2\beta_2\beta_3\beta_4^3\alpha_1^2 + \alpha_2^3\beta_2^2\alpha_1^2 + 2\alpha_2^3\beta_3^2\alpha_1^2 - 6\alpha_4^2\beta_1\beta_3^2\alpha_1^2 + 3\alpha_2\alpha_4\beta_2\beta_3^2\alpha_1^2 \\ & +\alpha_2\beta_2^2\beta_4^2\alpha_1^2 + 4\alpha_2\beta_3^2\beta_4^2\alpha_1^2 + 3\alpha_2^2\beta_1\beta_4^2\alpha_1^2 - 3\alpha_4\beta_1\beta_3\beta_4^2\alpha_1^2 - 2\alpha_2\beta_2\beta_3\beta_4^2\alpha_1^2 - 2\alpha_2^4\beta_1\alpha_1^2 \\ & +3\alpha_2\alpha_4\beta_2^2\beta_3\alpha_1^2 + 6\alpha_2^2\alpha_4\beta_1\beta_3\alpha_1^2 + 3\alpha_2^3\beta_2\beta_3\alpha_1^2 - 2\alpha_4\beta_3^3\beta_4\alpha_1^2 + 2\alpha_2^2\beta_2^2\beta_4\alpha_1^2 \\ & -4\alpha_2^2\beta_3^2\beta_4\alpha_1^2 - 3\alpha_4\beta_2\beta_3^2\beta_4\alpha_1^2 - \alpha_2^3\beta_1\beta_4\alpha_1^2 + 3\alpha_4\beta_2^2\beta_3\beta_4\alpha_1^2 + 3\alpha_2\alpha_4\beta_1\beta_3\beta_4\alpha_1^2 \\ & -\alpha_2^2\beta_2\beta_3\beta_4\alpha_1^2 + 2\alpha_2\alpha_4\beta_3^4\alpha_1\alpha_1^2 + 2\beta_1\beta_2\beta_4^4\alpha_1 - 3\beta_1\beta_3\beta_4^4\alpha_1 + \alpha_2^3\beta_3^3\alpha_1 - 4\alpha_4^2\beta_1\beta_3^3\alpha_1 \\ & +5\alpha_2\alpha_4\beta_2\beta_3^3\alpha_1 - \beta_3^3\beta_4^3\alpha_1 + \alpha_4\beta_1^2\beta_4^3\alpha_1 + 3\beta_2\beta_3^2\beta_4^3\alpha_1 + 3\alpha_2\beta_1\beta_2\beta_4^3\alpha_1 - \beta_2^2\beta_3\beta_4^3\alpha_1 \\ & +\alpha_2^3\alpha_4\beta_1^2\alpha_1 + 3\alpha_2\alpha_4\beta_2^2\beta_3^2\alpha_1 + 3\alpha_2^2\alpha_4\beta_1\beta_3^2\alpha_1 + 2\alpha_2^3\beta_2\beta_3^2\alpha_1 + 3\alpha_2\beta_3^3\beta_4^2\alpha_1 \\ & +3\alpha_2\alpha_4\beta_1^2\beta_4^2\alpha_1 - 6\alpha_4\beta_1\beta_3^2\beta_4^2\alpha_1 - \alpha_2\beta_2\beta_3^2\beta_4^2\alpha_1 - 3\alpha_2^2\beta_1\beta_2\beta_4^2\alpha_1 - \alpha_2\beta_2^2\beta_3\beta_4^2\alpha_1 \\ & +6\alpha_2^2\beta_1\beta_3\beta_4^2\alpha_1 - 3\alpha_2^4\beta_1\beta_2\alpha_1 + \alpha_2^3\beta_2^2\beta_3\alpha_1 - 3\alpha_2^4\beta_1\beta_3\alpha_1 - \alpha_4\beta_3^4\beta_4\alpha_1 - 3\alpha_2^2\beta_3^3\beta_4\alpha_1 \\ & +\alpha_4\beta_2\beta_3^3\beta_4\alpha_1 + 3\alpha_2^2\alpha_4\beta_1^2\beta_4\alpha_1 + 3\alpha_4\beta_2^2\beta_3^2\beta_4\alpha_1 - 3\alpha_2\alpha_4\beta_1\beta_3^2\beta_4\alpha_1 - 2\alpha_2^2\beta_2\beta_3^2\beta_4\alpha_1 \\ & -7\alpha_2^3\beta_1\beta_2\beta_4\alpha_1 + \alpha_2^2\beta_2^2\beta_3\beta_4\alpha_1 + \alpha_2\alpha_4\beta_3^5 - \alpha_4^2\beta_1\beta_3^4 + 2\alpha_2\alpha_4\beta_2\beta_3^4 - \beta_1\beta_2^2\beta_4^4 \\ & -2\beta_1\beta_3^2\beta_4^4 + 3\beta_1\beta_2\beta_3\beta_4^4 + \alpha_2\alpha_4\beta_2^2\beta_3^3 + \beta_2\beta_3^3\beta_4^3 - 4\alpha_2\beta_1\beta_2^2\beta_4^3 - \beta_2^2\beta_3^2\beta_4^3 - \alpha_2\beta_1\beta_3^2\beta_4^3 \\ & +\alpha_4\beta_1^2\beta_3\beta_4^3 + 7\alpha_2\beta_1\beta_2\beta_3\beta_4^3 - \alpha_2^4\beta_1\beta_2^2 - \alpha_2^4\beta_1\beta_3^2 + \alpha_2\beta_3^4\beta_4^2 - 3\alpha_4\beta_1\beta_3^3\beta_4^2 \\ & -\alpha_2\beta_2\beta_3^3\beta_4^2 - 6\alpha_2^2\beta_1\beta_2^2\beta_4^2 - 2\alpha_2\beta_2^2\beta_3^2\beta_4^2 + 3\alpha_2^2\beta_1\beta_3^2\beta_4^2 + 3\alpha_2\alpha_4\beta_1^2\beta_3\beta_4^2 + 3\alpha_2^2\beta_1\beta_2\beta_3\beta_4^2 \\ & +\alpha_2^3\alpha_4\beta_1^2\beta_3 - 2\alpha_2^4\beta_1\beta_2\beta_3 - \alpha_2^2\beta_3^4\beta_4 + \alpha_4\beta_2\beta_3^4\beta_4 + \alpha_4\beta_2^2\beta_3^3\beta_4 - 3\alpha_2\alpha_4\beta_1\beta_3^3\beta_4 \\ & -2\alpha_2^2\beta_2\beta_3^3\beta_4 - 4\alpha_2^3\beta_1\beta_2^2\beta_4 - \alpha_2^2\beta_2^2\beta_3^2\beta_4 + \alpha_2^3\beta_1\beta_3^2\beta_4 + 3\alpha_2^2\alpha_4\beta_1^2\beta_3\beta_4 - 3\alpha_2^3\beta_1\beta_2\beta_3\beta_4\Big) \\ & \Big/ \Big((\alpha_1+\beta_3)(2\alpha_1^2\beta_4^2 + 3\alpha_4\alpha_1^2\beta_3 + \alpha_2\alpha_1^2\beta_4 + 3\alpha_4\alpha_1\beta_3^2 - 2\alpha_1\beta_2\beta_4^2 + 3\alpha_1\beta_3\beta_4^2 \\ & \quad -2\alpha_2^2\alpha_1\beta_2 - 3\alpha_2^2\alpha_1\beta_3 - 4\alpha_2\alpha_1\beta_2\beta_4 + \alpha_4\beta_3^3 - 2\alpha_2^2\beta_3^2 + 3\alpha_2\beta_1\beta_4^2 + \alpha_2^3\beta_1 \\ & \quad -2\alpha_2^2\beta_2\beta_3 - \alpha_2\beta_3^2\beta_4 + 3\alpha_2^2\beta_1\beta_4 - 4\alpha_2\beta_2\beta_3\beta_4 + \alpha_4\alpha_1^3 - \alpha_1^2\alpha_2^2 + \beta_1\beta_4^3 + \beta_3^2\beta_4^2 \\ & \quad -2\beta_2\beta_3\beta_4^2)^{3/2}\Big). \end{aligned}$$

Respectively,

$$
\begin{aligned}
\alpha_{1,c} \;=\;& \tfrac{1}{(\alpha_2+\beta_4)^2}\Big(\alpha_2\beta_4+\alpha_4\beta_3+\alpha_1\alpha_4+\beta_4^2-\big(\alpha_1\beta_4^2+2\alpha_4\alpha_1\beta_3+\alpha_2\alpha_1\beta_4-\alpha_3\alpha_1\beta_4+\alpha_4\beta_3^2\\
&-\alpha_2^2\beta_3-\alpha_2\alpha_3\beta_3-\alpha_2\beta_3\beta_4-\alpha_3\beta_3\beta_4+\alpha_4\alpha_1^2-\alpha_1\alpha_2\alpha_3\big)^2\big/\big(2\alpha_1^2\beta_4^2+3\alpha_4\alpha_1^2\beta_3+3\alpha_2\alpha_1^2\beta_4\\
&-2\alpha_3\alpha_1^2\beta_4+3\alpha_4\alpha_1\beta_3^2+\alpha_1\beta_3\beta_4^2-\alpha_2^2\alpha_1\beta_3-4\alpha_2\alpha_3\alpha_1\beta_3-4\alpha_3\alpha_1\beta_3\beta_4+\alpha_4\beta_3^3\\
&-2\alpha_2^2\beta_3^2-2\alpha_2\alpha_3\beta_3^2+3\alpha_2\beta_1\beta_4^2+\alpha_2^3\beta_1-3\alpha_2\beta_3^2\beta_4-2\alpha_3\beta_3^2\beta_4+3\alpha_2^2\beta_1\beta_4+\alpha_4\alpha_1^3+\alpha_2^2\alpha_1^2\\
&-2\alpha_2\alpha_3\alpha_1^2+\beta_1\beta_4^3-\beta_3^2\beta_4^2)\Big);\\
\beta_{1,c} \;=\;& \Big(\beta_1\beta_3\alpha_2^5-\alpha_4\beta_1^2\alpha_2^4-\alpha_1\beta_3^2\alpha_2^4+\alpha_1\alpha_3\beta_1\alpha_2^4+\alpha_1^2\beta_3\alpha_2^4+2\alpha_3\beta_1\beta_3\alpha_2^4-\alpha_1\beta_1\beta_4\alpha_2^4\\
&+2\beta_1\beta_3\beta_4\alpha_2^4-2\alpha_1\alpha_3\beta_3^2\alpha_2^3-2\alpha_1\beta_1\beta_4^2\alpha_2^3+\alpha_1^3\alpha_3\alpha_2^3+\alpha_1\alpha_3^2\beta_1\alpha_2^3-3\alpha_1^2\alpha_4\beta_1\alpha_2^3\\
&-\alpha_1^2\alpha_3\beta_3\alpha_2^3+\alpha_3^2\beta_1\beta_3\alpha_2^3-3\alpha_1\alpha_4\beta_1\beta_3\alpha_2^3-\alpha_1^3\beta_4\alpha_2^3+\beta_3^3\beta_4\alpha_2^3-4\alpha_4\beta_1^2\beta_4\alpha_2^3-3\alpha_1\beta_3^2\beta_4\alpha_2^3\\
&+\alpha_1\alpha_3\beta_1\beta_4\alpha_2^3+3\alpha_1^2\beta_3\beta_4\alpha_2^3+5\alpha_3\beta_1\beta_3\beta_4\alpha_2^3-\alpha_4\beta_3^4\alpha_2^2+\alpha_1\alpha_4\beta_3^3\alpha_2^2-2\beta_1\beta_3\beta_4^3\alpha_2^2\\
&-\alpha_1^3\alpha_3^2\alpha_2^2-\alpha_1\alpha_3^2\beta_3^2\alpha_2^2+3\alpha_1^2\alpha_4\beta_3^2\alpha_2^2-2\alpha_1^3\beta_4^2\alpha_2^2+2\beta_3^3\beta_4^2\alpha_2^2-6\alpha_4\beta_1^2\beta_4^2\alpha_2^2-4\alpha_1\beta_3^2\beta_4^2\alpha_2^2\\
&-3\alpha_1\alpha_3\beta_1\beta_4^2\alpha_2^2+4\alpha_1^2\beta_3\beta_4^2\alpha_2^2+3\alpha_3\beta_1\beta_3\beta_4^2\alpha_2^2-2\alpha_1^4\alpha_4\alpha_2^2-2\alpha_1^2\alpha_3^2\beta_3\alpha_2^2-\alpha_1^3\alpha_4\beta_3\alpha_2^2\\
&+2\alpha_3\beta_3^3\beta_4\alpha_2^2-2\alpha_1\alpha_3\beta_3^2\beta_4\alpha_2^2+3\alpha_4\beta_1\beta_3^2\beta_4\alpha_2^2+3\alpha_1^3\alpha_3\beta_4\alpha_2^2+3\alpha_1\alpha_3^2\beta_1\beta_4\alpha_2^2\\
&-6\alpha_1^2\alpha_4\beta_1\beta_4\alpha_2^2-\alpha_1^2\alpha_3\beta_3\beta_4\alpha_2^2+3\alpha_3^2\beta_1\beta_3\beta_4\alpha_2^2-3\alpha_1\alpha_4\beta_1\beta_3\beta_4\alpha_2^2-2\alpha_3\alpha_4\beta_3^4\alpha_2\\
&+2\alpha_1\beta_1\beta_4^4\alpha_2-\beta_1\beta_3\beta_4^4\alpha_2-3\alpha_1\alpha_3\alpha_4\beta_3^3\alpha_2+\alpha_4^2\beta_1\beta_3^3\alpha_2-\alpha_1^3\beta_4^3\alpha_2+\beta_3^3\beta_4^3\alpha_2-4\alpha_4\beta_1^2\beta_4^3\alpha_2\\
&-\alpha_1\alpha_2\beta_4^3(3\beta_3^2+5\alpha_3\beta_1)+3\alpha_1^2\beta_3\beta_4^3\alpha_2-\alpha_3\beta_1\beta_3\beta_4^3\alpha_2+3\alpha_1^2\alpha_3\alpha_4\beta_3^2\alpha_2+3\alpha_1\alpha_4^2\beta_1\beta_3^2\alpha_2\\
&+3\alpha_3\beta_3^3\beta_4^2\alpha_2-\alpha_1\alpha_3\beta_3^2\beta_4^2\alpha_2+6\alpha_4\beta_1\beta_3^2\beta_4^2\alpha_2+2\alpha_1^3\alpha_3\beta_4^2\alpha_2+3\alpha_1\alpha_3^2\beta_1\beta_4^2\alpha_2\\
&-3\alpha_1^2\alpha_4\beta_1\beta_4^2\alpha_2-2\alpha_1^2\alpha_3\beta_3\beta_4^2\alpha_2+3\alpha_3^2\beta_1\beta_3\beta_4^2\alpha_2+3\alpha_1\alpha_4\beta_1\beta_3\beta_4^2\alpha_2+3\alpha_1^4\alpha_3\alpha_4\alpha_2\\
&+\alpha_1^3\alpha_4^2\beta_1\alpha_2+7\alpha_1^3\alpha_3\alpha_4\beta_3\alpha_2+3\alpha_1^2\alpha_4^2\beta_1\beta_3\alpha_2-3\alpha_4\beta_3^4\beta_4\alpha_2+\alpha_3^2\beta_3^3\beta_4\alpha_2-\alpha_1^3\alpha_3^2\beta_4\alpha_2\\
&+\alpha_1\alpha_3^2\beta_3^2\beta_4\alpha_2+6\alpha_1^2\alpha_4\beta_3^2\beta_4\alpha_2-3\alpha_1^4\alpha_4\beta_4\alpha_2-\alpha_1^2\alpha_3^2\beta_3\beta_4\alpha_2+\alpha_1\beta_1\beta_4^5-\alpha_3^2\alpha_4\beta_3^4\\
&-\alpha_4\beta_1^2\beta_4^4-\alpha_1\beta_3^2\beta_4^4-2\alpha_1\alpha_3\beta_1\beta_4^4+\alpha_1^2\beta_3\beta_4^4-\alpha_3\beta_1\beta_3\beta_4^4-4\alpha_1\alpha_3^2\alpha_4\beta_3^3+\alpha_3\beta_3^3\beta_4^3\\
&-\alpha_1\alpha_3\beta_3^2\beta_4^3+3\alpha_4\beta_1\beta_3^2\beta_4^3+\alpha_1\alpha_3^2\beta_1\beta_4^3-2\alpha_1^2\alpha_3\beta_3\beta_4^3+\alpha_3^2\beta_1\beta_3\beta_4^3+3\alpha_1\alpha_4\beta_1\beta_3\beta_4^3\\
&-6\alpha_1^2\alpha_3^2\alpha_4\beta_3^2-2\alpha_4\beta_3^4\beta_4^2+\alpha_3^2\beta_3^3\beta_4^2-\alpha_1\alpha_4\beta_3^3\beta_4^2+2\alpha_1\alpha_3^2\beta_3^2\beta_4^2+3\alpha_1^2\alpha_4\beta_3^2\beta_4^2-\alpha_1^4\alpha_4\beta_4^2\\
&+\alpha_1^2\beta_3\beta_4^2(\alpha_3^2+\alpha_1\alpha_4)-\alpha_1^4\alpha_3^2\alpha_4-4\alpha_1^3\alpha_3^2\alpha_4\beta_3-3\alpha_3\alpha_4\beta_3^4\beta_4-7\alpha_1\alpha_3\alpha_4\beta_3^3\beta_4+\alpha_4^2\beta_1\beta_3^3\beta_4
\end{aligned}
$$

$$
\begin{aligned}
&-3\alpha_1\alpha_4\beta_3^2\beta_4(\alpha_1\alpha_3-\alpha_4\beta_1)+2\alpha_1^4\alpha_3\alpha_4\beta_4+\alpha_1^3\alpha_4^2\beta_1\beta_4+3\alpha_1^2\alpha_4\beta_3\beta_4(\alpha_1\alpha_3+\alpha_4\beta_1)\Big)\Big/\\
&\Big((\alpha_2+\beta_4)\big(2\alpha_1^2\beta_4^2+3\alpha_4\alpha_1^2\beta_3+3\alpha_2\alpha_1^2\beta_4-2\alpha_3\alpha_1^2\beta_4+3\alpha_4\alpha_1\beta_3^2+\alpha_1\beta_3\beta_4^2-\alpha_2^2\alpha_1\beta_3\\
&-4\alpha_2\alpha_3\alpha_1\beta_3-4\alpha_3\alpha_1\beta_3\beta_4+\alpha_4\beta_3^3-2\alpha_2^2\beta_3^2-2\alpha_2\alpha_3\beta_3^2+3\alpha_2\beta_1\beta_4^2+\alpha_2^3\beta_1-3\alpha_2\beta_3^2\beta_4\\
&-2\alpha_3\beta_3^2\beta_4+3\alpha_2^2\beta_1\beta_4+\alpha_4\alpha_1^3+\alpha_2^2\alpha_1^2-2\alpha_2\alpha_3\alpha_1^2+\beta_1\beta_4^3-\beta_3^2\beta_4^2\big)^{\frac{3}{2}}\Big).
\end{aligned}
$$

If $\alpha'_4 = 0$, then

$$\begin{pmatrix} 1 & 0 \\ \alpha''_1 & 2\alpha''_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \alpha'_1 & 2\alpha'_2 \end{pmatrix} g^{-1}$$

is valid.

Therefore, in $\alpha'_2 \neq 0$ (respectively, $\alpha'_2 \neq 0$) case, due to Lemma 2.2, for a canonical MSC we take

$$A_3 = P'(A')A'(P'(A')^{-1})^{\otimes 2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_{1,c} & \lambda & 1 & -1 \end{pmatrix},$$

where $P'(A') = \begin{pmatrix} 1 & 0 \\ \alpha'_1 & 2\alpha'_2 \end{pmatrix}$,

$$
\begin{aligned}
\beta_{1,c} \;=\;& -\frac{\big(\alpha_1\beta_3+\alpha_2\beta_1+\alpha_1^2+\beta_1\beta_4\big)\big(2\alpha_1\beta_2+3\alpha_1\beta_3-3\alpha_2\beta_1+\alpha_1^2+2\beta_3^2+2\beta_2\beta_3-3\beta_1\beta_4\big)}{4(\alpha_1+\beta_3)^4}\\
&-\frac{\beta_1\big(\alpha_1^2\beta_4-\alpha_2\alpha_1\beta_2-\alpha_2\alpha_1\beta_3-\alpha_1\beta_2\beta_4+\alpha_1\beta_3\beta_4-\alpha_2\beta_3^2+\alpha_2^2\beta_1-\alpha_2\beta_2\beta_3+2\alpha_2\beta_1\beta_4+\beta_1\beta_4^2-\beta_2\beta_3\beta_4\big)}{(\alpha_1+\beta_3)^4}.
\end{aligned}
$$

Respectively,

$$\begin{array}{rcl}\beta_{1,c} & = & -\frac{\left(\alpha_2\beta_4+\alpha_4\beta_3+\alpha_1\alpha_4+\beta_4^2\right)\left(3\alpha_2\beta_4-3\alpha_4\beta_3+2\alpha_3\beta_4+2\alpha_2^2+2\alpha_3\alpha_2-3\alpha_1\alpha_4+\beta_4^2\right)}{4(\alpha_2+\beta_4)^4} \\ & & -\frac{\alpha_4\left(\alpha_1\beta_4^2+2\alpha_4\alpha_1\beta_3+\alpha_2\alpha_1\beta_4-\alpha_3\alpha_1\beta_4+\alpha_4\beta_3^2-\alpha_2^2\beta_3-\alpha_2\alpha_3\beta_3-\alpha_2\beta_3\beta_4-\alpha_3\beta_3\beta_4+\alpha_4\alpha_1^2-\alpha_1\alpha_2\alpha_3\right)}{(\alpha_2+\beta_4)^4}.\end{array}$$

If $\alpha_4' = 0, \alpha_2' = 0$, then

$$\begin{array}{rcl}\alpha_1'' & = & \alpha_1', \\ \alpha_2'' & = & 0, \\ \alpha_4'' & = & 0, \\ \beta_1'' & = & \frac{\beta_1'+(1+\lambda-3\alpha_1')\xi_2}{\eta_2};\end{array}$$

and in the cases $1+\lambda-3\alpha_1' \neq 0$, i.e., $\lambda-\alpha_1' \neq 2\alpha_1'-1$, or $1+\lambda-3\alpha_1' = 0, \beta_1' = 0$, one can make $\beta_1'' = 0$ to get

$$A_4 = \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 0 \\ 0 & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 0 \end{pmatrix},$$

where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_3} + \beta_1\frac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2}$.
Respectively, $\alpha_{1,c} = \frac{\beta_4}{(\alpha_2+\beta_4)} + \alpha_4\frac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2})$.

If $1+\lambda-3\alpha_1' = 0$ and $\beta_1 \neq 0$,
Respectively, $\alpha_4 \neq 0$, we can make $\beta_1'' = 1$ to get

$$A_5 = \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 0 \\ 1 & 2\alpha_{1,c}-1 & 1-\alpha_{1,c} & 0 \end{pmatrix},$$

where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_3} + \beta_1\frac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2}$.
Respectively, $\alpha_{1,c} = \frac{\beta_4}{(\alpha_2+\beta_4)} + \alpha_4\frac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2})$.

Consider $\mathfrak{S}_3$. In this case $Tr_1(A) = (0,0) = (\alpha_1+\beta_3, \alpha_2+\beta_4)$ and $Tr_2(A) = (\alpha_1+\beta_2, \alpha_3+\beta_4) \neq (0,0)$. Here also we discuss in detail only $\alpha_1+\beta_2 \neq 0$ case and $\alpha_3+\beta_4 \neq 0$ case is represented in parentheses. At $g = \begin{pmatrix} \alpha_1+\beta_2 & \alpha_3+\beta_4 \\ 0 & 1 \end{pmatrix}$ (respectively, $g = \begin{pmatrix} \alpha_1+\beta_2 & \alpha_3+\beta_4 \\ 1 & 0 \end{pmatrix}$) we have

$$A' = gA(g^{-1})^{\otimes 2} = \begin{pmatrix} \alpha_1' & \alpha_2' & \alpha_2' & \alpha_4' \\ \beta_1' & 1-\alpha_1' & -\alpha_1' & -\alpha_2' \end{pmatrix}, \text{ where}$$

$$\begin{array}{rcl}\alpha_1' & = & \frac{\alpha_1}{\alpha_1+\beta_2} + \beta_1\frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2}, \\ \alpha_2' & = & \frac{2\alpha_2\alpha_1\beta_2+\alpha_3\beta_2^2-\alpha_2^2\beta_1-\alpha_3^2\beta_1+2\alpha_2\alpha_3\beta_1+2\alpha_2\alpha_1^2-\alpha_3\alpha_1^2}{(\alpha_1+\beta_2)^2} \\ \alpha_4' & = & -\frac{\beta_1(\alpha_2-\alpha_3)^3}{(\alpha_1+\beta_2)^2} + \frac{3\alpha_1(\alpha_2-\alpha_3)^2}{\alpha_1+\beta_2} + \alpha_4\beta_2 + \alpha_2^2 - 2\alpha_3^2 + \alpha_2\alpha_3 + \alpha_1\alpha_4, \\ \beta_1' & = & \frac{\beta_1}{(\alpha_1+\beta_2)^2}.\end{array}$$

Respectively,

$$\begin{array}{rcl}\alpha_1' & = & \frac{\beta_4}{\alpha_3+\beta_4} + \alpha_4\frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2}, \\ \alpha_2' & = & \frac{1}{(\alpha_3+\beta_4)^2}(\alpha_3^2\beta_2 + 2\alpha_3\beta_3\beta_4 - \alpha_4\beta_2^2 - \alpha_4\beta_3^2 + 2\alpha_4\beta_2\beta_3 - \beta_2\beta_4^2 + 2\beta_3\beta_4^2), \\ \alpha_4' & = & -\frac{\alpha_4(\beta_3-\beta_2)^3}{(\alpha_3+\beta_4)^2} + \frac{3\beta_4(\beta_3-\beta_2)^2}{\alpha_3+\beta_4} + \alpha_3\beta_1 - 2\beta_2^2 + \beta_3^2 + \beta_2\beta_3 + \beta_1\beta_4, \\ \beta_1' & = & \frac{\alpha_4}{(\alpha_3+\beta_4)^2}).\end{array}$$

If $g^{-1}=\begin{pmatrix} 1 & 0\\ \xi_2 & \eta_2\end{pmatrix}$, then for $A''=gA'(g^{-1})^{\otimes 2}$ one has

$$\begin{array}{rcl} \alpha_1'' &=& \alpha_1'+2\alpha_2'\xi_2+\alpha_4'\xi_2^2,\\ \alpha_2'' &=& (\alpha_2'+\alpha_4'\xi_2)\eta_2,\\ \alpha_4'' &=& \alpha_4'\eta_2^2,\\ \beta_1'' &=& \frac{\beta_1'+\xi_2-3\alpha_1'\xi_2-3\alpha_2'\xi_2^2-\alpha_4\xi_2^3}{\eta_2}. \end{array}$$

The last relations are the same that are (1) but only in $\lambda=0$ case. Therefore, in a similar manner one gets:

If $\alpha_4'\neq 0$ (respectively, $\alpha_4'\neq 0$) it can be reduced to

$$A_6=\begin{pmatrix} \alpha_{1,c} & 0 & 0 & 1\\ \beta_{1,c} & 1-\alpha_{1,c} & -\alpha_{1,c} & 0\end{pmatrix}\simeq\begin{pmatrix} \alpha_{1,c} & 0 & 0 & 1\\ -\beta_{1,c} & 1-\alpha_{1,c} & -\alpha_{1,c} & 0\end{pmatrix},\ \text{where}$$

$$\begin{array}{rcl} \alpha_{1,c} &=& (2\alpha_4\alpha_1^2\beta_2+\alpha_4\alpha_1\beta_2^2-\alpha_2\alpha_4\alpha_1\beta_1+\alpha_3\alpha_4\alpha_1\beta_1+\alpha_2^2\alpha_1\beta_2-3\alpha_2\alpha_3\alpha_1\beta_2-\alpha_3^2\beta_2^2-\alpha_2^3\beta_1\\ && -\alpha_2\alpha_3^2\beta_1+2\alpha_2^2\alpha_3\beta_1-\alpha_2\alpha_4\beta_1\beta_2+\alpha_3\alpha_4\beta_1\beta_2+\alpha_4\alpha_1^3-\alpha_1^2\alpha_2\alpha_3)\big/(3\alpha_4\alpha_1^2\beta_2\\ && +3\alpha_4\alpha_1\beta_2^2+5\alpha_2^2\alpha_1\beta_2-\alpha_3^2\alpha_1\beta_2-4\alpha_2\alpha_3\alpha_1\beta_2+\alpha_4\beta_2^3+\alpha_2^2\beta_2^2-2\alpha_3^2\beta_2^2+\alpha_2\alpha_3\beta_2^2\\ && -\alpha_2^3\beta_1+\alpha_3^3\beta_1-3\alpha_2\alpha_3^2\beta_1+3\alpha_2^2\alpha_3\beta_1+\alpha_4\alpha_1^3+4\alpha_2^2\alpha_1^2+\alpha_3^2\alpha_1^2-5\alpha_2\alpha_3\alpha_1^2), \end{array}$$

$$\begin{array}{rcl} \beta_{1,c} &=& (\alpha_4^2\alpha_1^3\beta_1+6\alpha_2\alpha_4\alpha_1^3\beta_2-\alpha_3\alpha_4\alpha_1^3\beta_2+3\alpha_3\alpha_4\alpha_1^2\beta_2^2-3\alpha_3^2\alpha_4\alpha_1^2\beta_1+3\alpha_2\alpha_3\alpha_4\alpha_1^2\beta_1\\ && -4\alpha_2^3\alpha_1^2\beta_2+\alpha_3^3\alpha_1^2\beta_2-3\alpha_2\alpha_3^2\alpha_1^2\beta_2+3\alpha_4^2\alpha_1^2\beta_1\beta_2-2\alpha_2\alpha_4\alpha_1\beta_2^3+\alpha_3\alpha_4\alpha_1\beta_2^3\\ && -2\alpha_2^3\alpha_1\beta_2^2-\alpha_3^3\alpha_1\beta_2^2-3\alpha_2^2\alpha_3\alpha_1\beta_2^2+3\alpha_4^2\alpha_1\beta_1\beta_2^2+4\alpha_2^4\alpha_1\beta_1+2\alpha_2\alpha_3^3\alpha_1\beta_1\\ && -6\alpha_2^3\alpha_3\alpha_1\beta_1+3\alpha_2^2\alpha_4\alpha_1\beta_1\beta_2-3\alpha_3^2\alpha_4\alpha_1\beta_1\beta_2-\alpha_3\alpha_4\beta_2^4-\alpha_2\alpha_3^2\beta_2^3-\alpha_2^2\alpha_3\beta_2^3\\ && +\alpha_4^2\beta_1\beta_2^3+\alpha_2^3\alpha_4\beta_1^2-\alpha_3^3\alpha_4\beta_1^2+3\alpha_2\alpha_3^2\alpha_4\beta_1^2-3\alpha_2^2\alpha_3\alpha_4\beta_1^2+3\alpha_2^2\alpha_4\beta_1\beta_2^2\\ && -3\alpha_2\alpha_3\alpha_4\beta_1\beta_2^2+2\alpha_2^4\beta_1\beta_2+\alpha_3^4\beta_1\beta_2+\alpha_2\alpha_3^3\beta_1\beta_2-3\alpha_2^2\alpha_3^2\beta_1\beta_2-\alpha_2^3\alpha_3\beta_1\beta_2\\ && +4\alpha_2\alpha_4\alpha_1^4-2\alpha_3\alpha_4\alpha_1^4+2\alpha_2\alpha_3^2\alpha_1^3-4\alpha_2^2\alpha_3\alpha_1^3)\big/\big(3\alpha_4\alpha_1^2\beta_2+3\alpha_4\alpha_1\beta_2^2+5\alpha_2^2\alpha_1\beta_2\\ && -\alpha_3^2\alpha_1\beta_2-4\alpha_2\alpha_3\alpha_1\beta_2+\alpha_4\beta_2^3+\alpha_2^2\beta_2^2-2\alpha_3^2\beta_2^2+\alpha_2\alpha_3\beta_2^2-\alpha_2^3\beta_1+\alpha_3^3\beta_1-3\alpha_2\alpha_3^2\beta_1\\ && +3\alpha_2^2\alpha_3\beta_1+\alpha_4\alpha_1^3+4\alpha_2^2\alpha_1^2+\alpha_3^2\alpha_1^2-5\alpha_2\alpha_3\alpha_1^2\big)^{3/2}. \end{array}$$

Respectively,

$$\begin{array}{rcl} \alpha_{1,c} &=& \big(-\alpha_4\beta_3^3+2\alpha_4\beta_2\beta_3^2+\alpha_3\beta_4\beta_3^2-\alpha_4\beta_2^2\beta_3-\alpha_3\alpha_4\beta_1\beta_3-\alpha_4\beta_1\beta_4\beta_3-3\alpha_3\beta_2\beta_4\beta_3\\ && -\alpha_3^2\beta_2^2+2\alpha_3\beta_1\beta_4^2+\alpha_3\alpha_4\beta_1\beta_2+\alpha_3^2\beta_1\beta_4+\alpha_4\beta_1\beta_2\beta_4-\beta_2\beta_4^2\beta_3+\beta_1\beta_4^3\big)\big/\big(\alpha_3^3\beta_1\\ && -2\alpha_3^2\beta_2^2+\alpha_3^2\beta_3^2+\alpha_3^2\beta_2\beta_3+3\alpha_3^2\beta_1\beta_4+3\alpha_3\beta_1\beta_4^2-\alpha_3\beta_2^2\beta_4+5\alpha_3\beta_3^2\beta_4-4\alpha_3\beta_2\beta_3\beta_4\\ && +\alpha_4\beta_2^3-\alpha_4\beta_3^3+3\alpha_4\beta_2\beta_3^2-3\alpha_4\beta_2^2\beta_3+\beta_1\beta_4^3+\beta_2^2\beta_4^2+4\beta_3^2\beta_4^2-5\beta_2\beta_3\beta_4^2\big), \end{array}$$

$$\begin{array}{rcl} \beta_{1,c} &=& \big(\alpha_3^4\left(-\beta_1\right)\beta_2+\alpha_4\alpha_3^3\beta_1^2-\alpha_3^3\beta_2\beta_3^2-\alpha_3^3\beta_2^2\beta_3+\alpha_3^3\beta_1\beta_2\beta_4-2\alpha_3^3\beta_1\beta_3\beta_4+3\alpha_4\alpha_3^2\beta_1\beta_3^2\\ && +3\alpha_3^2\beta_1\beta_2\beta_4^2-3\alpha_4\alpha_3^2\beta_1\beta_2\beta_3-\alpha_3^2\beta_2^3\beta_4-2\alpha_3^2\beta_3^3\beta_4+3\alpha_4\alpha_3^2\beta_1^2\beta_4-3\alpha_3^2\beta_2\beta_3^2\beta_4\\ && +\alpha_4\alpha_3\beta_2^4+2\alpha_4\alpha_3\beta_3^4-\alpha_4\alpha_3\beta_2\beta_3^3-\alpha_3\beta_1\beta_2\beta_4^3+6\alpha_3\beta_1\beta_3\beta_4^3-3\alpha_4\alpha_3\beta_2^2\beta_3^2+\alpha_3\beta_2^3\beta_4^2\\ && -4\alpha_3\beta_3^3\beta_4^2+3\alpha_4\alpha_3\beta_1^2\beta_4^2-3\alpha_3\beta_2^2\beta_3\beta_4^2+\alpha_4\alpha_3\beta_2^3\beta_3-3\alpha_4\alpha_3\beta_1\beta_2^2\beta_4+3\alpha_4\alpha_3\beta_1\beta_3^2\beta_4\\ && -\alpha_4^2\beta_1\beta_2^3+\alpha_4^2\beta_1\beta_3^3+\alpha_4\beta_1^2\beta_4^3-3\alpha_4^2\beta_1\beta_2\beta_3^2-3\alpha_4\beta_1\beta_2^2\beta_4^2+3\alpha_4\beta_1\beta_2\beta_3\beta_4^2\\ && +3\alpha_4^2\beta_1\beta_2^2\beta_3+4\alpha_4\beta_3^4\beta_4-6\alpha_4\beta_2\beta_3^3\beta_4+2\alpha_4\beta_2^3\beta_3\beta_4-2\beta_1\beta_2\beta_4^4+4\beta_1\beta_3\beta_4^4\\ && -4\beta_2\beta_3^2\beta_4^3\alpha_1^2+2\beta_2^2\beta_3\beta_4^3\big)\big(\alpha_3^3\beta_1-2\alpha_3^2\beta_2^2+\alpha_3^2\beta_3^2+\alpha_3^2\beta_2\beta_3+3\alpha_3^2\beta_1\beta_4+3\alpha_3\beta_1\beta_4^2\\ && -\alpha_3\beta_2^2\beta_4+5\alpha_3\beta_3^2\beta_4-4\alpha_3\beta_2\beta_3\beta_4+\alpha_4\beta_2^3-\alpha_4\beta_3^3+3\alpha_4\beta_2\beta_3^2-3\alpha_4\beta_2^2\beta_3\\ && +\beta_1\beta_4^3+\beta_2^2\beta_4^2+4\beta_3^2\beta_4^2-5\beta_2\beta_3\beta_4^2\big)^{3/2}\big) \end{array}$$

with the separating system $\alpha_{1,c},\beta_{1,c}^2$.

If $\alpha'_4 = 0$ and $\alpha'_2 \neq 0$ (respectively, $\alpha'_4 = 0$, $\alpha'_2 \neq 0$), then A is reduced to

$$A_7 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_{1,c} & 1 & 0 & -1 \end{pmatrix},$$

where
$\beta_{1,c} = \frac{-3\alpha_1^2\beta_2^2+4\alpha_2\alpha_1^2\beta_1-2\alpha_1\beta_2^3+6\alpha_2\alpha_1\beta_1\beta_2+2\alpha_3\alpha_1\beta_1\beta_2-\alpha_2^2\beta_1^2-\alpha_3^2\beta_1^2+2\alpha_2\alpha_3\beta_1^2+2\alpha_2\beta_1\beta_2^2+2\alpha_3\beta_1\beta_2^2+\alpha_1^4}{4(\alpha_1+\beta_2)^4}.$
Respectively,
$\beta_{1,c} = \frac{-3\alpha_3^2\beta_4^2+4\alpha_4\beta_3\beta_4^2-2\alpha_3^3\beta_4+2\alpha_3\alpha_4\beta_2\beta_4+6\alpha_3\alpha_4\beta_3\beta_4-\alpha_4^2\beta_2^2-\alpha_4^2\beta_3^2+2\alpha_3^2\alpha_4\beta_2+2\alpha_3^2\alpha_4\beta_3+2\alpha_4^2\beta_2\beta_3+\beta_4^4}{4(\alpha_3+\beta_4)^4}$
is a separating system.

If $\alpha'_4 = \alpha'_2 = 0$ and $1 - 3\alpha'_1 \neq 0$ or $1 - 3\alpha'_1 = 0$ and $\beta'_1 = 0$, (respectively, $1 - 3\alpha'_1 \neq 0$ or $1 - 3\alpha'_1 = 0$ and $\beta'_1 = 0$,), then A is reduced to

$$A_8 = \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 0 \\ 0 & 1 - \alpha_{1,c} & -\alpha_{1,c} & 0 \end{pmatrix},$$

where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_2} + \beta_1 \frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2}$ (respectively, $\alpha_{1,c} = \frac{\beta_4}{\alpha_3+\beta_4} + \alpha_4 \frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2}$).

If $\alpha'_4 = \alpha'_2 = 0$, $1 - 3\alpha'_1 = 0$ and $\beta_1 \neq 0$ (respectively, $1 - 3\alpha'_1 = 0$, $\alpha_4 \neq 0$) case one comes to

$$A_9 = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}.$$

Finally, let us consider $\mathfrak{S}_4$. In this case,

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1 & -\alpha_2 \end{pmatrix},\ g^{-1} = \begin{pmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{pmatrix} \text{and } A' = \begin{pmatrix} \alpha'_1 & \alpha'_2 & \alpha'_2 & \alpha'_4 \\ \beta'_1 & -\alpha'_1 & -\alpha'_1 & -\alpha'_2 \end{pmatrix}$$

such that

$$\begin{aligned} \alpha'_1 &= \frac{1}{\Delta(g)} \left(-\beta_1\eta_1\xi_1^2 + \alpha_1\eta_2\xi_1^2 + 2\alpha_1\eta_1\xi_1\xi_2 + 2\alpha_2\eta_2\xi_1\xi_2 + \alpha_2\eta_1\xi_2^2 + \alpha_4\eta_2\xi_2^2\right), \\ \alpha'_2 &= \frac{-1}{\Delta(g)} \left(\beta_1\eta_1^2\xi_1 - 2\alpha_1\eta_1\eta_2\xi_1 - \alpha_2\eta_2^2\xi_1 - \alpha_1\eta_1^2\xi_2 - 2\alpha_2\eta_1\eta_2\xi_2 - \alpha_4\eta_2^2\xi_2\right), \\ \alpha'_4 &= \frac{-1}{\Delta(g)} \left(\beta_1\eta_1^3 - 3\alpha_1\eta_1^2\eta_2 - 3\alpha_2\eta_1\eta_2^2 - \alpha_4\eta_2^3\right), \\ \beta'_1 &= \frac{1}{\Delta(g)} \left(\beta_1\xi_1^3 - 3\alpha_1\xi_1^2\xi_2 - 3\alpha_2\xi_1\xi_2^2 - \alpha_4\xi_2^3\right), \end{aligned}$$

where $\Delta(g) = \xi_1\eta_2 - \xi_2\eta_1$.

If $\alpha_4 \neq 0$ and $p_A(t) = \beta_1 - 3\alpha_1 t - 3\alpha_2 t^2 - \alpha_4 t^3$ has an unique root λ_1, that is if $\alpha_2^2 = \alpha_1\alpha_4$, $\alpha_2^3 = -\beta_1\alpha_4^2$, then at $g^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{1}{\sqrt{\alpha_4}} & \lambda_1 \end{pmatrix}$ one gets $A' = gA(g^{-1})^{\otimes 2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = A_{12}$.

If $p_A(t) = \beta_1 - 3\alpha_1 t - 3\alpha_2 t^2 - \alpha_4 t^3$ has two distinct roots λ_1, λ_2 then $\alpha_2^2 \neq \alpha_1\alpha_4$ and if the multiplicity of λ_2 is two then by puting $\xi_2 = \lambda_1\xi_1, \eta_2 = \lambda_2\eta_1$, where $\xi_1 \neq 0, \eta_1 \neq 0$, one gets
$\beta'_1 = \alpha'_4 = \alpha'_2 = 0$ as far as $\alpha'_2 = \frac{\xi_1\eta_1^2}{\Delta(g)}(-\beta_1 + 2\alpha_1\lambda_2 + \alpha_2\lambda_2^2 + \lambda_1(\alpha_1 + 2\alpha_2\lambda_2 + \alpha_4\lambda_2^2))$,
$\alpha_1 + 2\alpha_2\lambda_2 + \alpha_4\lambda_2^2 = 0$ and

$$0 = -\beta_1 + 3\alpha_1\lambda_2 + 3\alpha_2\lambda_2^2 + \alpha_4\lambda_2^3 = -\beta_1 + 2\alpha_1\lambda_2 + \alpha_2\lambda_2^2 + \lambda_2(\alpha_1 + 2\alpha_2\lambda_2 + \alpha_4\lambda_2^2).$$

In this case, $\lambda_2(\alpha_4\lambda_1^2 + 2\alpha_2\lambda_1 + \alpha_1) - \beta_1 + 2\alpha_1\lambda_1 + \alpha_2\lambda_1^2 \neq 0$, $\Delta = \xi_1\eta_1(\lambda_2 - \lambda_1)$. Therefore, by taking

$$g^{-1} = \begin{pmatrix} (\lambda_2 - \lambda_1)(\lambda_2(\alpha_4\lambda_1^2 + 2\alpha_2\lambda_1 + \alpha_1) - \beta_1 + 2\alpha_1\lambda_1 + \alpha_2\lambda_1^2)^{-1} & 1 \\ \lambda_1(\lambda_2 - \lambda_1)(\lambda_2(\alpha_4\lambda_1^2 + 2\alpha_2\lambda_1 + \alpha_1) - \beta_1 + 2\alpha_1\lambda_1 + \alpha_2\lambda_1^2)^{-1} & \lambda_2 \end{pmatrix},$$

one gets

$$A' = gA(g^{-1})^{\otimes 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \simeq A_{10}.$$

If $p_A(t) = \beta_1 - 3\alpha_1 t - 3\alpha_2 t^2 - \alpha_4 t^3$ has three distinct roots $\lambda_1, \lambda_2, \lambda_3$, then by taking

$$g^{-1} = \begin{pmatrix} 0 & 1 \\ (\lambda_2 - \lambda_1)(\lambda_2(\alpha_4\lambda_1^2 + 2\alpha_2\lambda_1 + \alpha_1) - \beta_1 + 2\alpha_1\lambda_1 + \alpha_2\lambda_1^2)^{-1} & \lambda_2 \end{pmatrix},$$

we get

$$A' = gA(g^{-1})^{\otimes 2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \simeq A_{10}.$$

Now we consider $\alpha_4 = 0$ case. Let us take g with $\eta_1 = 0$ to have $\alpha_4' = 0$. In this case $\Delta = \xi_1\eta_2$ and

$$\begin{aligned} \alpha_1' &= \xi_1\left(\alpha_1 + 2\alpha_2\tfrac{\xi_2}{\xi_1}\right), \\ \alpha_2' &= \alpha_2\eta_2, \\ \beta_1' &= \tfrac{\xi_1^2}{\eta_2}\left(\beta_1 - 3\alpha_1\tfrac{\xi_2}{\xi_1} - 3\alpha_2(\tfrac{\xi_2}{\xi_1})^2\right). \end{aligned}$$

Case a. Let $\alpha_2 \neq 0$. Consider $\frac{\xi_2}{\xi_1} = \frac{-\alpha_1}{2\alpha_2}$ to get $\alpha_1' = 0, \alpha_2' = 1$ and $\beta_1' = \xi_1^2\frac{3\alpha_1^2+4\alpha_2\beta_1}{4}$. Therefore, one can make β_1' equal to 0 or 1, depending on $3\alpha_1^2 + 4\alpha_2\beta_1$ to be zero or not we have

$$A_{10} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \text{ or } A_{11} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \text{ respectively.}$$

Case b. Let $\alpha_2 = 0$. Then $\alpha_2' = \alpha_4' = 0$ and $\alpha_1' = \xi_1\alpha_1, \beta_1' = \frac{\xi_1^2}{\eta_2}\left(\beta_1 - 3\alpha_1\frac{\xi_2}{\xi_1}\right)$. Therefore, if $\alpha_1 \neq 0$ one can make $\alpha_1' = 1, \beta_1' = 0$ to get $A' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}$, which is isomorphic to A_{10}, if $\alpha_1 = 0$ then $\alpha_1' = 0$ and one can make $\beta_1' = 1$ to come to

$$A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

□

Note that in A_9, A_{10}, A_{11} and A_{12} cases there are no rational invariants except for constants.

Here are the corresponding theorems in $Char(\mathbb{F}) = 2, 3$ cases without proofs. For space matter in formulations of the results for entries of the canonical matrices we refer to the corresponding expressions in Theorem 3.2, wherever it is possible. Some expressions in these formulations intentionally are not simplified for easy understanding in referring to Theorem 3.2.

Theorem 3.2. *Let $Char(\mathbb{F}) = 2$. If $\Delta(A) \neq 0$ then*

$$A \simeq \begin{pmatrix} \alpha_{1,c} & \alpha_{2,c} & \alpha_{2,c}+1 & \alpha_{4,c} \\ \beta_{1,c} & -\alpha_{1,c} & 1-\alpha_{1,c} & -\alpha_{2,c} \end{pmatrix} = A_{1,2},$$

where $\alpha_{1,c}, \alpha_{2,c}, \alpha_{4,c}$ and $\beta_{1,c}$ are as in Theorem 3.2.

If $\Delta(A) = 0$, $\alpha_1 + \beta_3 \neq 0$ $(\alpha_2 + \beta_4 \neq 0)$ and

$$\begin{aligned}\alpha_{4,2}(A) &= -2\alpha_1^2\beta_4^2\lambda - 4\alpha_2\alpha_1^2\beta_4\lambda - 4\alpha_1\beta_3\beta_4^2\lambda - 4\alpha_2^2\alpha_1\beta_3\lambda - 8\alpha_2\alpha_1\beta_3\beta_4\lambda - 2\alpha_2^2\beta_3^2\lambda \\ &\quad -4\alpha_2\beta_3^2\beta_4\lambda + 4\alpha_1^2\beta_4^2 + 3\alpha_4\alpha_1^2\beta_3 + 5\alpha_2\alpha_1^2\beta_4 + 3\alpha_4\alpha_1\beta_3^2 + 5\alpha_1\beta_3\beta_4^2 - \alpha_2^2\alpha_1\beta_3 \\ &\quad +4\alpha_2\alpha_1\beta_3\beta_4 + \alpha_4\beta_3^3 - 2\alpha_2^2\beta_3^2 + 3\alpha_2\beta_1\beta_4^2 + \alpha_2^3\beta_1 - \alpha_2\beta_3^2\beta_4 + 3\alpha_2^2\beta_1\beta_4 - 2\alpha_2^2\alpha_1^2\lambda \\ &\quad +\alpha_4\alpha_1^3 + \alpha_2^2\alpha_1^2 - 2\beta_3^2\beta_4^2\lambda + \beta_1\beta_4^3 + \beta_3^2\beta_4^2 - \alpha_4\alpha_1^2\beta_3 + \alpha_2\alpha_1^2\beta_4 + \alpha_4\alpha_1\beta_3^2 \\ &\quad +\alpha_1\beta_3\beta_4^2 - \alpha_2^2\alpha_1\beta_3 + \alpha_4\beta_3^3 + \alpha_2\beta_1\beta_4^2 + \alpha_2^3\beta_1 - \alpha_2\beta_3^2\beta_4 + \alpha_2^2\beta_1\beta_4 + \alpha_4\alpha_1^3 \\ &\quad +\alpha_2^2\alpha_1^2 + \beta_1\beta_4^3 + \beta_3^2\beta_4^2 \neq 0,\end{aligned}$$

then

$$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 1 \\ \beta_{1,c} & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 0 \end{pmatrix} = A_{2,2},$$

where $\alpha_{1,c}$ *and* $\beta_{1,c}$ *are as in Theorem 3.2.*

If $\Delta(A) = 0$*,* $\alpha_1+\beta_3 \neq 0$ $(\alpha_2+\beta_4 \neq 0)$*,* $\alpha_{4,2}(A) = 0$ *and*

$$\begin{aligned}\alpha_{2,2}(A) &= -\alpha_1^2\beta_4 + \alpha_2\alpha_1\beta_2 + \alpha_2\alpha_1\beta_3 + \alpha_1\beta_2\beta_4 - \alpha_1\beta_3\beta_4 + \alpha_2\beta_3^2 - \alpha_2^2\beta_1 + \alpha_2\beta_2\beta_3 \\ &\quad -2\alpha_2\beta_1\beta_4 - \beta_1\beta_4^2 + \beta_2\beta_3\beta_4 = -\alpha_1^2\beta_4 + \alpha_2\alpha_1\beta_2 + \alpha_2\alpha_1\beta_3 + \alpha_1\beta_2\beta_4 - \alpha_1\beta_3\beta_4 \\ &\quad +\alpha_2\beta_3^2 - \alpha_2^2\beta_1 + \alpha_2\beta_2\beta_3 - \beta_1\beta_4^2 + \beta_2\beta_3\beta_4 \neq 0;\end{aligned}$$

respectively,

$$\begin{aligned}\alpha'_{2,2}(A) &= -\alpha_1\beta_4^2 - 2\alpha_4\alpha_1\beta_3 - \alpha_2\alpha_1\beta_4 + \alpha_3\alpha_1\beta_4 - \alpha_4\beta_3^2 + \alpha_2^2\beta_3 + \alpha_2\alpha_3\beta_3 + \alpha_2\beta_3\beta_4 \\ &\quad +\alpha_3\beta_3\beta_4 - \alpha_4\alpha_1^2 + \alpha_2\alpha_3\alpha_1 = -\alpha_1\beta_4^2 - \alpha_2\alpha_1\beta_4 + \alpha_3\alpha_1\beta_4 - \alpha_4\beta_3^2 + \alpha_2^2\beta_3 \\ &\quad +\alpha_2\alpha_3\beta_3 + \alpha_2\beta_3\beta_4 + \alpha_3\beta_3\beta_4 - \alpha_4\alpha_1^2 + \alpha_2\alpha_3\alpha_1 \neq 0,\end{aligned}$$

then

$$A \simeq \begin{pmatrix} \alpha_{1,c} & 1 & 1 & 0 \\ 0 & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 1 \end{pmatrix} = A_{3,2},$$

where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_3} + \beta_1\frac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2}$ $\Big($*respectively,* $\alpha_{1,c} = \frac{\beta_4}{(\alpha_2+\beta_4)} + \alpha_4\frac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2}\Big)$.

If $\Delta(A) = 0$*,* $\alpha_1+\beta_3 \neq 0$ $(\alpha_2+\beta_4 \neq 0)$ $\alpha_{4,2}(A) = 0$ *and* $\alpha_{2,2}(A) = 0$*,*

$$\beta_{1,2}(A) = 1 + \frac{2\alpha_1+\beta_2}{\alpha_1+\beta_3} + \beta_1\frac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2} \neq 0 \textit{ or } \beta_{1,2}(A) = 0,\ \beta_1 = 0$$

$\Big($*respectively,* $\alpha'_{2,2}(A) = 0$, $\beta'_{1,2}(A) = 1 + \frac{\alpha_3+2\beta_4}{\alpha_2+\beta_4} + \alpha_4\frac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2} \neq 0$ *or* $\beta'_{1,2}(A) = 0,\ \alpha_4 = 0\Big)$

then

$$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 0 \\ 0 & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 0 \end{pmatrix} = A_{4,2},$$

where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_3} + \beta_1\frac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2}$ $\Big($*respectively,* $\alpha_{1,c} = \frac{\beta_4}{(\alpha_2+\beta_4)} + \alpha_4\frac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2}\Big)$.

If $\Delta(A) = 0$*,* $\alpha_1+\beta_3 \neq 0$ $(\alpha_2+\beta_4 \neq 0)$ $\alpha_{4,2}(A) = 0$ *and* $\alpha_{2,2}(A) = 0$*,* $\beta_{1,2}(A) = 0$*,* $\beta_1 \neq 0$ $\Big($*respectively,* $\alpha'_{2,2}(A) = 0$*,* $\beta'_{1,2}(A) = 0$*,* $\alpha_4 \neq 0\Big)$

then

$$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 0 \\ 1 & 1 & 1-\alpha_{1,c} & 0 \end{pmatrix} = A_{5,2},$$

where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_3} + \beta_1\frac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2}$, $\Big($*respectively,* $\alpha_{1,c} = \frac{\beta_4}{\alpha_2+\beta_4} + \alpha_4\frac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2}\Big)$.

If $Tr_1(A) = (0,0) = (\alpha_1+\beta_3, \alpha_2+\beta_4)$*,* $\alpha_1+\beta_2 \neq 0$ $(\ \alpha_3+\beta_4 \neq 0)$ *and*

$$\begin{aligned}\alpha_{4,3}(A) &= -\tfrac{\beta_1(\alpha_2-\alpha_3)^3}{(\alpha_1+\beta_2)^2} + \tfrac{3\alpha_1(\alpha_2-\alpha_3)^2}{\alpha_1+\beta_2} + \alpha_4\beta_2 - 2\alpha_3^2 + \alpha_2^2 + \alpha_2\alpha_3 + \alpha_1\alpha_4 \\ &= -\tfrac{\beta_1(\alpha_2-\alpha_3)^3}{(\alpha_1+\beta_2)^2} + \tfrac{\alpha_1(\alpha_2-\alpha_3)^2}{\alpha_1+\beta_2} + \alpha_4\beta_2 + \alpha_2^2 + \alpha_2\alpha_3 + \alpha_1\alpha_4 \neq 0\end{aligned}$$

(*respectively,*

$$\begin{array}{rcl}\alpha'_{4,3}(A) & = & -\frac{\alpha_4(\beta_3-\beta_2)^3}{(\alpha_3+\beta_4)^2}+\frac{3\beta_4(\beta_3-\beta_2)^2}{\alpha_3+\beta_4}+\alpha_3\beta_1-2\beta_2^2+\beta_3^2+\beta_2\beta_3+\beta_1\beta_4 \\ & = & -\frac{\alpha_4(\beta_3-\beta_2)^3}{(\alpha_3+\beta_4)^2}+\frac{\beta_4(\beta_3-\beta_2)^2}{\alpha_3+\beta_4}+\alpha_3\beta_1+\beta_3^2+\beta_2\beta_3+\beta_1\beta_4\neq 0\big)\end{array}$$

then

$$A\simeq\begin{pmatrix}\alpha_{1,c} & 0 & 0 & 1\\ \beta_{1,c} & 1-\alpha_{1,c} & \lambda-\alpha_{1,c} & 0\end{pmatrix}=A_{6,2},$$

where $\alpha_{1,c}$ *and* $\beta_{1,c}$ *are as in Theorem 3.2.*

If $Tr_1(A)=(0,0)=(\alpha_1+\beta_3,\alpha_2+\beta_4)$, $\alpha_1+\beta_2\neq 0$ ($\alpha_3+\beta_4\neq 0)$, $\alpha_{4,3}(A)=0$ *and*

$$\begin{array}{rcl}\alpha_{2,3}(A) & = & 2\alpha_2\alpha_1\beta_2+\alpha_3\beta_2^2-\alpha_2^2\beta_1-\alpha_3^2\beta_1+2\alpha_2\alpha_3\beta_1+2\alpha_2\alpha_1^2-\alpha_3\alpha_1^2\\ & = & \alpha_3\beta_2^2-\alpha_2^2\beta_1-\alpha_3^2\beta_1-\alpha_3\alpha_1^2\neq 0\end{array}$$

(*respectively,* $\alpha'_{2,3}(A)=\alpha_3^2\beta_2+2\alpha_3\beta_3\beta_4-\alpha_4\beta_2^2-\alpha_4\beta_3^2+2\alpha_4\beta_2\beta_3-\beta_2\beta_4^2+2\beta_3\beta_4^2=\alpha_3^2\beta_2$ $-\alpha_4\beta_2^2-\alpha_4\beta_3^2-\beta_2\beta_4^2\neq 0\big)$

then

$$A\simeq\begin{pmatrix}\alpha_{1,c} & 1 & 1 & 0\\ 0 & 1-\alpha_{1,c} & -\alpha_{1,c} & -1\end{pmatrix}=A_{7,2},$$

where $\alpha'_1 = \frac{\alpha_1}{\alpha_1+\beta_2}+\beta_1\frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2}$
(*respectively,* $\alpha'_1 = \frac{\beta_4}{\alpha_3+\beta_4}+\alpha_4\frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2}$) *and* $\alpha_{1,c}$ *separates such non-isomorphic algebras.*

If $Tr_1(A)=(0,0)=(\alpha_1+\beta_3,\alpha_2+\beta_4)$, $\alpha_1+\beta_2\neq 0$ ($\alpha_3+\beta_4\neq 0)$ *and* $\alpha_{4,3}(A)=0$, $\alpha_{2,3}(A)=0$ *and* $1-(\frac{\alpha_1}{\alpha_1+\beta_2}+\beta_1\frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2})\neq 0$ *or* $1-(\frac{\alpha_1}{\alpha_1+\beta_2}+\beta_1\frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2})=0$ *and* $\beta_1=0$
(*respectively,* $\alpha'_{4,3}(A)=0$, $\alpha'_{2,3}(A)=0$ *and* $1-(\frac{\beta_4}{\alpha_3+\beta_4}+\alpha_4\frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2})\neq 0$ *or*
$1-(\frac{\beta_4}{\alpha_3+\beta_4}+\alpha_4\frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2})=0$ *and* $\alpha_4=0\big)$

then

$$A\simeq\begin{pmatrix}\alpha_{1,c} & 0 & 0 & 0\\ 0 & 1-\alpha_{1,c} & \alpha_{1,c} & 0\end{pmatrix}=A_{8,2},$$

where $\alpha_{1,c}=\frac{\alpha_1}{\alpha_1+\beta_2}+\beta_1\frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2}$ (*respectively,* $\alpha_{1,c}=\frac{\beta_4}{\alpha_3+\beta_4}+\alpha_4\frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2}$).

If $Tr_1(A)=(0,0)=(\alpha_1+\beta_3,\alpha_2+\beta_4)$, $\alpha_1+\beta_2\neq 0$ ($\alpha_3+\beta_4\neq 0)$ *and*
$\alpha_{4,3}(A)=0$, $\alpha_{2,3}(A)=0$, $1-(\frac{\alpha_1}{\alpha_1+\beta_2}+\beta_1\frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2})=0$ *and* $\beta_1\neq 0$
(*respectively,* $\alpha'_{4,3}(A)=0$, $\alpha'_{2,3}(A)=0$, $1-(\frac{\beta_4}{\alpha_3+\beta_4}+\alpha_4\frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2})=0$ *and* $\alpha_4\neq 0\big)$

then

$$A\simeq\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 0 & 1 & 0\end{pmatrix}=A_{9,2}.$$

If $Tr_1(A)=Tr_2(A)=0,\alpha_1=0,\alpha_2=0,\alpha_4=0,\beta_1=0$

then

$$A\simeq\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\text{ is the trivial algebra.}$$

If $Tr_1(A) = Tr_2(A) = 0$, $\alpha_2^2 = \alpha_1\alpha_4,\ \alpha_2^3 = -\beta_1\alpha_4^2$ *and* $\alpha_4 \neq 0$ *or* $\alpha_4 = \alpha_2 = \alpha_1 = 0$ *and* $\beta_1 \neq 0$

then

$$A \simeq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = A_{12,2}.$$

If $Tr_1(A) = Tr_2(A) = 0$ *and* $\alpha_4 \neq 0$, $\alpha_2^3 = -\beta_1\alpha_4^2$, $\alpha_2^2 \neq \alpha_1\alpha_4$ *or* $\alpha_4 \neq 0$, $\alpha_2^3 \neq -\beta_1\alpha_4^2$ *or* $\alpha_4 = 0$ *and* $\alpha_2 \neq 0$, $\alpha_1 = 0$ *or* $\alpha_4 = 0, \alpha_2 = 0, \alpha_1 \neq 0$

then

$$A \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = A_{10,2}.$$

If $Tr_1(A) = Tr_2(A) = 0$, $\alpha_4 = 0$, $\alpha_2 \neq 0$ *and* $\alpha_1 \neq 0$

then

$$A \simeq \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} = A_{11,2}.$$

Theorem 3.3. *Let* $Char.(\mathbb{F}) = 3$. *If* $\Delta(A) \neq 0$ *then*

$$A \simeq \begin{pmatrix} \alpha_{1,c} & \alpha_{2,c} & \alpha_{2,c}+1 & \alpha_{4,c} \\ \beta_{1,c} & -\alpha_{1,c} & 1-\alpha_{1,c} & -\alpha_{2,c} \end{pmatrix} = A_{1,3},$$

where $\alpha_{1,c}, \alpha_{2,c}, \alpha_{4,c}$ *and* $\beta_{1,c}$ *are as in Theorem 3.2.*

If $\Delta(A) = 0$, $\alpha_1 + \beta_3 \neq 0$ $(\alpha_2 + \beta_4 \neq 0)$ *and*

$$\begin{array}{rcl} \alpha_{4,2}(A) & = & -2\alpha_1^2\beta_4^2\lambda - 4\alpha_2\alpha_1^2\beta_4\lambda - 4\alpha_1\beta_3\beta_4^2\lambda - 4\alpha_2^2\alpha_1\beta_3\lambda - 8\alpha_2\alpha_1\beta_3\beta_4\lambda - 2\alpha_2^2\beta_3^2\lambda \\ & & -4\alpha_2\beta_3^2\beta_4\lambda + 4\alpha_1^2\beta_4^2 + 3\alpha_4\alpha_1^2\beta_3 + 5\alpha_2\alpha_1^2\beta_4 + 3\alpha_4\alpha_1\beta_3^2 + 5\alpha_1\beta_3\beta_4^2 - \alpha_2^2\alpha_1\beta_3 \\ & & +4\alpha_2\alpha_1\beta_3\beta_4 + \alpha_4\beta_3^3 - 2\alpha_2^2\beta_3^2 + 3\alpha_2\beta_1\beta_4^2 + \alpha_2^3\beta_1 - \alpha_2\beta_3^2\beta_4 + 3\alpha_2^2\beta_1\beta_4 - 2\alpha_2^2\alpha_1^2\lambda \\ & & +\alpha_4\alpha_1^3 + \alpha_2^2\alpha_1^2 - 2\beta_3^2\beta_4^2\lambda + \beta_1\beta_4^3 + \beta_3^2\beta_4^2 = -2\alpha_1^2\beta_4^2\lambda - \alpha_2\alpha_1^2\beta_4\lambda - \alpha_1\beta_3\beta_4^2\lambda \\ & & -\alpha_2^2\alpha_1\beta_3\lambda - 2\alpha_2\alpha_1\beta_3\beta_4\lambda - 2\alpha_2^2\beta_3^2\lambda - \alpha_2\beta_3^2\beta_4\lambda + \alpha_1^2\beta_4^2 + 2\alpha_2\alpha_1^2\beta_4 + 2\alpha_1\beta_3\beta_4^2 \\ & & -\alpha_2^2\alpha_1\beta_3 + \alpha_2\alpha_1\beta_3\beta_4 + \alpha_4\beta_3^3 - 2\alpha_2^2\beta_3^2 + \alpha_2^3\beta_1 - \alpha_2\beta_3^2\beta_4 - 2\alpha_2^2\alpha_1^2\lambda + \alpha_4\alpha_1^3 \\ & & +\alpha_2^2\alpha_1^2 - 2\beta_3^2\beta_4^2\lambda + \beta_1\beta_4^3 + \beta_3^2\beta_4^2 \neq 0 \end{array}$$

then

$$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 1 \\ \beta_{1,c} & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 0 \end{pmatrix} \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 1 \\ -\beta_{1,c} & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 0 \end{pmatrix} = A_{2,3},$$

where $\alpha_{1,c}$ *and* $\beta_{1,c}$ *are as in Theorem 3.2.*

If $\Delta(A) = 0$, $\alpha_1 + \beta_3 \neq 0$ $\big(\alpha_2 + \beta_4 \neq 0\big)$ $\alpha_{4,2}(A) = 0$ *and*

$$\begin{array}{rcl} \alpha_{2,2}(A) & = & -\alpha_1^2\beta_4 + \alpha_2\alpha_1\beta_2 + \alpha_2\alpha_1\beta_3 + \alpha_1\beta_2\beta_4 - \alpha_1\beta_3\beta_4 + \alpha_2\beta_3^2 - \alpha_2^2\beta_1 + \alpha_2\beta_2\beta_3 \\ & & -2\alpha_2\beta_1\beta_4 - \beta_1\beta_4^2 + \beta_2\beta_3\beta_4 \neq 0 \end{array}$$

$\big($*respectively,*

$$\begin{array}{rcl} \alpha'_{2,2}(A) & = & -\alpha_1\beta_4^2 - 2\alpha_4\alpha_1\beta_3 - \alpha_2\alpha_1\beta_4 + \alpha_3\alpha_1\beta_4 - \alpha_4\beta_3^2 + \alpha_2^2\beta_3 + \alpha_2\alpha_3\beta_3 + \alpha_2\beta_3\beta_4 \\ & & +\alpha_3\beta_3\beta_4 - \alpha_4\alpha_1^2 + \alpha_2\alpha_3\alpha_1 \neq 0\big) \end{array}$$

then

$$A \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_{1,c} & \lambda & 1 & -1 \end{pmatrix} = A_{3,3},$$

where $\beta_{1,c}$ *are as in Theorem 3.2.*

If $\Delta(A) = 0$, $\alpha_1+\beta_3 \neq 0$ $(\alpha_2+\beta_4 \neq 0)$, $\alpha_{4,2}(A) = 0$, $\alpha_{2,2}(A) = 0$ $\big($*respectively,* $\alpha'_{2,2}(A) = 0\big)$ *and* $\lambda \neq 2$ *or* $\lambda = 2$ *and* $\beta_1 = 0$ (*respectively,* $\alpha_4 = 0$) *then*

$$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 0 \\ 0 & \lambda-\alpha_{1,c} & 1-\alpha_{1,c} & 0 \end{pmatrix} = A_{4,3},$$

where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_3} + \beta_1\frac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2}$ (*respectively,* $\alpha_{1,c} = \frac{\beta_4}{(\alpha_2+\beta_4)} + \alpha_4\frac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2}$).

If $\Delta(A) = 0$, $\alpha_1+\beta_3 \neq 0$ $(\alpha_2+\beta_4 \neq 0)$, $\alpha_{4,2}(A) = 0$, $\alpha_{2,2}(A) = 0$ $\big($*respectively,* $\alpha'_{2,2}(A) = 0\big)$ *and* $\lambda = 2$ *and* $\beta_1 \neq 0$ (*respectively,* $\alpha_4 \neq 0$) *then*

$$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 0 \\ 1 & 2-\alpha_{1,c} & 1-\alpha_{1,c} & 0 \end{pmatrix} = A_{5,3},$$

where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_3} + \beta_1\frac{\alpha_2+\beta_4}{(\alpha_1+\beta_3)^2}$ (*respectively,* $\alpha_{1,c} = \frac{\beta_4}{(\alpha_2+\beta_4)} + \alpha_4\frac{\alpha_1+\beta_3}{(\alpha_2+\beta_4)^2}$).

If $Tr_1(A) = (0,0) = (\alpha_1+\beta_3, \alpha_2+\beta_4)$, $\alpha_1+\beta_2 \neq 0$ ($\alpha_3+\beta_4 \neq 0$) *and*

$$\alpha_{4,3}(A) = -\frac{\beta_1(\alpha_2-\alpha_3)^3}{(\alpha_1+\beta_2)^2} + \alpha_4\beta_2 - 2\alpha_3^2 + \alpha_2^2 + \alpha_2\alpha_3 + \alpha_1\alpha_4 \neq 0$$

$\big($*respectively,* $\alpha'_{4,3}(A) = -\frac{\alpha_4(\beta_3-\beta_2)^3}{(\alpha_3+\beta_4)^2} + \alpha_3\beta_1 - 2\beta_2^2 + \beta_3^2 + \beta_2\beta_3 + \beta_1\beta_4 \neq 0\big)$ *then*

$$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 1 \\ \beta_{1,c} & 1-\alpha_{1,c} & -\alpha_{1,c} & 0 \end{pmatrix} \approx \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 1 \\ -\beta_{1,c} & 1-\alpha_{1,c} & -\alpha_{1,c} & 0 \end{pmatrix} = A_{6,3},$$

where $\alpha_{1,c}$ *and* $\beta_{1,c}$ *are as in Theorem 3.2.*

If $Tr_1(A) = (0,0) = (\alpha_1+\beta_3, \alpha_2+\beta_4)$, $\alpha_1+\beta_2 \neq 0$ ($\alpha_3+\beta_4 \neq 0$) *and* $\alpha_{4,3}(A) = 0$, $\alpha_{2,3}(A) = 2\alpha_2\alpha_1\beta_2 + \alpha_3\beta_2^2 - \alpha_2^2\beta_1 - \alpha_3^2\beta_1 + 2\alpha_2\alpha_3\beta_1 + 2\alpha_2\alpha_1^2 - \alpha_3\alpha_1^2 \neq 0$ $\big($*respectively,* $\alpha'_{4,3}(A) = 0$, $\alpha'_{2,3}(A) = \alpha_3^2\beta_2 + 2\alpha_3\beta_3\beta_4 - \alpha_4\beta_2^2 - \alpha_4\beta_3^2 + 2\alpha_4\beta_2\beta_3 - \beta_2\beta_4^2 + 2\beta_3\beta_4^2 \neq 0\big)$ *then*

$$A \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_{1,c} & 1 & 0 & -1 \end{pmatrix} = A_{7,3},$$

where $\beta_{1,c} = \frac{\alpha_2\alpha_1^2\beta_1 - 2\alpha_1\beta_2^3 + 2\alpha_3\alpha_1\beta_1\beta_2 - \alpha_2^2\beta_1^2 - \alpha_3^2\beta_1^2 + 2\alpha_2\alpha_3\beta_1^2 + 2\alpha_2\beta_1\beta_2^2 + 2\alpha_3\beta_1\beta_2^2 + \alpha_1^4}{(\alpha_1+\beta_2)^4}$ $\big($*respectively,* $\beta_{1,c} = \frac{\alpha_4\beta_3\beta_4^2 - 2\alpha_3^3\beta_4 + 2\alpha_3\alpha_4\beta_2\beta_4 - \alpha_4^2\beta_2^2 - \alpha_4^2\beta_3^2 + 2\alpha_3^2\alpha_4\beta_2 + 2\alpha_3^2\alpha_4\beta_3 + 2\alpha_4^2\beta_2\beta_3 + \beta_4^4}{(\alpha_3+\beta_4)^4}\big)$.

If $Tr_1(A) = (0,0) = (\alpha_1+\beta_3, \alpha_2+\beta_4)$, $\alpha_1+\beta_2 \neq 0$ ($\alpha_3+\beta_4 \neq 0$) *and* $\alpha_{4,3}(A) = 0$, $\alpha_{2,3}(A) = 0$ $\big($*respectively,* $\alpha'_{4,3}(A) = 0$, $\alpha'_{2,3}(A) = 0\big)$ *then*

$$A \simeq \begin{pmatrix} \alpha_{1,c} & 0 & 0 & 0 \\ 0 & 1-\alpha_{1,c} & \alpha_{1,c} & 0 \end{pmatrix} = A_{8,3},$$

where $\alpha_{1,c} = \frac{\alpha_1}{\alpha_1+\beta_2} + \beta_1\frac{\alpha_3-\alpha_2}{(\alpha_1+\beta_2)^2}$ $\big($*respectively,* $\alpha_{1,c} = \frac{\beta_4}{\alpha_3+\beta_4} + \alpha_4\frac{\beta_2-\beta_3}{(\alpha_3+\beta_4)^2}\big)$.

If $Tr_1(A) = Tr_2(A) = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_4 = 0, \beta_1 = 0$ *then*

$$A \simeq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is the trivial algebra.}$$

If $Tr_1(A) = Tr_2(A) = 0$ *and* $\alpha_4 \neq 0$ *or* $\alpha_4 = \alpha_1 = \alpha_2 = 0$, $\beta_1 \neq 0$ *then*

$$A \simeq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = A_{12,3}.$$

If $Tr_1(A) = Tr_2(A) = 0$, $\alpha_4 = 0$, $\alpha_2 \neq 0$ *and* $\beta_1 \neq 0$ *then*

$$A \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} = A_{9,3}.$$

If $Tr_1(A) = Tr_2(A) = 0$, $\alpha_4 = 0$ *and* $\alpha_2 \neq 0$, $\beta_1 = 0$ *or* $\alpha_4 = \alpha_2 = 0$, $\alpha_1 \neq 0$, $\beta_1 = 0$ *then*

$$A \simeq \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = A_{10,3}.$$

If $Tr_1(A) = Tr_2(A) = 0$, $\alpha_4 = 0$ *and* $\alpha_1 \neq 0, \alpha_2 = 0$, $\beta_1 \neq 0$ *then*

$$A \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{pmatrix} = A_{11,3}.$$

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